JYVÄSKYLÄ STUDIES IN COMPUTING
96

Nikolay V. Kuznetsov

Stability and Oscillations of Dynamical Systems

Theory and Applications

JYVÄSKYLÄN YLIOPISTO

JYVÄSKYLÄ STUDIES IN COMPUTING 96

Nikolay V. Kuznetsov

Stability and Oscillations of Dynamical Systems

Theory and Applications

Esitetään Jyväskylän yliopiston informaatioteknologian tiedekunnan suostumuksella
julkisesti tarkastettavaksi yliopiston Mattilanniemen salissa MaC102
joulukuun 12. päivänä 2008 kello 12.

Academic dissertation to be publicly discussed, by permission of
the Faculty of Information Technology of the University of Jyväskylä,
in Mattilanniemi, Auditorium MaC102, on December 12, 2008 at 12 o'clock noon.

UNIVERSITY OF JYVÄSKYLÄ

JYVÄSKYLÄ 2008

Stability and Oscillations of Dynamical Systems

Theory and Applications

JYVÄSKYLÄ STUDIES IN COMPUTING 96

Nikolay V. Kuznetsov

Stability and Oscillations of Dynamical Systems

Theory and Applications

UNIVERSITY OF JYVÄSKYLÄ

JYVÄSKYLÄ 2008

Editors
Timo Männikkö
Department of Mathematical Information Technology, University of Jyväskylä
Pekka Olsbo, Marja-Leena Tynkkynen
Publishing Unit, University Library of Jyväskylä

URN:ISBN:978-951-39-3798-0
ISBN 978-951-39-3798-0 (PDF)

ISBN 978-951-39-3428-6 (nid.)
ISSN 1456-5390

Jyväskylä University Printing House, Jyväskylä 2008

ABSTRACT

Kuznetsov, Nikolay V.
Stability and Oscillations of Dynamical Systems: Theory and Applications
Jyväskylä: University of Jyväskylä, 2008, 116 p.
(Jyväskylä Studies in Computing
ISSN 1456-5390; 96)
ISBN 978-951-39-3798-0 (PDF), 978-951-39-3428-6 (nid.)
Finnish summary
Diss.

The present work is devoted to questions of qualitative theory of discrete and continuous dynamical systems and its applications.

In the first Chapter, following the works of A.M. Lyapunov, O. Perron, and N.G. Chetaev, the problem of justifying the method of investigation of stability and instability by the first approximation of discrete and continuous dynamical systems is considered. A classical problem of stability by the first approximation of time-varying motions is completely proved in the general case. The Perron effects of sign reversal of characteristic exponent in solutions of the original system and the first approximation system with the same initial data are considered.

In the second Chapter, in the case when there exist two purely imaginary eigenvalues of the system of the first approximation, the qualitative behavior of two-dimensional autonomous systems is considered. Here, by the classical works of A. Poincare and A.M. Lyapunov, the method for the calculation of Lyapunov quantities, which define a qualitative behavior (winding or unwinding) of trajectories in the plane, is used. A new method for computing Lyapunov quantities, developed for the Euclidian coordinates and in the time domain and not requiring a transformation to normal form, is obtained and applied. The advantages of this method are due to its ideological simplicity and visualization power. The general formulas of the third Lyapunov quantity expressed in terms of the coefficients of the original system are obtained with the help of modern software tools of symbolic computation.

The content of the third Chapter is the application of qualitative theory to differential equations for the study of mathematical models of phase synchronization such as the systems of connected pendulums in the Huygens problem and the control systems by frequency of phase-locked loops.

Keywords: Lyapunov stability, instability, time-varying linearization, first approximation, Lyapunov exponents, Perron effects, Lyapunov quantity, focus value, limit cycles, symbolic computation, phase-locked loop, synchronization

Author Nikolay Kuznetsov
Department of Mathematical Information Technology,
University of Jyväskylä,
Finland

Supervisors Professor Pekka Neittaanmäki
Department of Mathematical Information Technology,
University of Jyväskylä,
Finland

Professor Gennady Leonov
Department of Applied Cybernetics,
Mathematics and Mechanics Faculty,
St.Petersburg State University,
Russia

Reviewers Professor Jan Awrejcewicz
Department of Automatics and Biomechanics,
Faculty of Mechanical Engineering,
Technical University of Łódź,
Poland

Professor Mihails Zakrževskis
Faculty of Transport and Mechanical Engineering,
Riga Technical University,
Latvia

Opponent Professor Sergei Abramovich
School of Education and Professional Studies,
State University of New York at Potsdam,
USA

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisors Prof. Pekka Neittaanmäki and Prof. Gennady Leonov for their guidance and continuous support. I appreciate the opportunity to work in Department of Mathematical Information Technology of the University of Jyväskylä very much.

The thesis work was mainly funded by Academy of Finland and Graduate School in Computing and Mathematical Science (COMAS). This research work would not have been possible without the graduate school.

I want as well thank professor Timo Eirola for his valuable comments and would like to thank reviewers of the thesis Prof. Awrejcewicz and Prof. Zakrževskis.

Also, I would like to thank all my colleagues and friends, for having believed in me in my effort and the support they have given me during my doctoral studies in the University of Jyväskylä.

I would like to extend my deepest thanks to my wife, Olga Kuznetsova, for her love and inspiration, for the joyful shining of her eyes and smile.

Finally, I would like to express my deepest appreciation to my parents, Elmira Gurmuzova and Vladimir Kuznetsov, for all the love and support I have received throughout my life.

LIST OF FIGURES

CONTENTS

INTRODUCTION AND STRUCTURE OF THE STUDY

The present work is devoted to questions of qualitative theory of discrete and continuous dynamical systems and its applications.

In the first Chapter, following the works of A.M. Lyapunov, O. Perron, and N.G. Chetaev, the problem of justifying the method of investigation of stability and instability by the first approximation of discrete and continuous dynamical systems is considered. A classical problem of stability by the first approximation of time-varying motions is completely proved in the general case. The material of this Chapter is based on the survey "Time-Varying Linearization and the Perron effects", written jointly with G.A. Leonov for International Journal of Bifurcation and Chaos in 2007. The original results of the author, represented in the paper, were reported at different international conferences [Kuznetsov & Leonov, 2005[3]; Kuznetsov & Leonov, 2005[4]] and published in the works [Kuznetsov & Leonov, 2005[1]; Kuznetsov & Leonov, 2005[2]; Kuznetsov *et al.,* 2006[1]]. In these works the solutions of problems, posed by the second supervisor, are obtained by the author. The first Chapter contains the following author's results: criteria of stability by the first approximation for regular and nonregular linearizations for discrete systems, two examples of discrete systems with the Perron effect of sign reversal of characteristic exponent of solutions of the original system and the first approximation system with the same initial data, criteria of stability and instability of solutions cascade (the case of uniform stability with respect to initial data), criteria of instability by Krasovsky for discrete and continuous systems, and criteria of instability by Lyapunov for discrete systems.

The problems, discussed in the first Chapter, are of importance for the development of the theory itself as well as for applications. The latter is connected to the fact that, at present, many specialists in chaotic dynamics believe that the positiveness of the largest characteristic exponent of linear system of the first approximation implies the instability of solutions of the original system. Moreover, there are a great number of computer experiments, in which various numerical methods for calculating characteristic exponents and Lyapunov exponents of linear system of the first approximation are applied. As a rule, the authors ignore the justification of the linearization procedure and use the numerical values of exponents so obtained to construct various numerical characteristics of attractors of the original nonlinear systems.

In the second Chapter, in the case when there exist two purely imaginary eigenvalues of the system of the first approximation, the qualitative behavior of two-dimensional autonomous systems is considered. Here, by the classical works of A. Poincare and A.M. Lyapunov, the method for the calculation of Lyapunov quantities, which define a qualitative behavior (winding or unwinding) of trajectories in the plane, is used. A new method for computing Lyapunov quantities, developed for the Euclidian coordinates and in the time domain and not requiring a transformation to normal form, is obtained and applied. The advantages of this method are due to its ideological simplicity and visualization power. The general formulas of the third Lyapunov quantity expressed in terms of the coefficients of the original system are obtained with the help of modern software tools of symbolic computation (MatLab and Maple). The material of the second Chapter is based on the works [Kuznetsov & Leonov, 2007; Kuznetsov *et al.,* 2008[2]; Leonov *et al.*, 2008; Kuznetsov & Leonov, 2008[1]; Kuznetsov & Leonov, 2008[2]], in which the formulation of problem is due to the supervisors and a computational procedure, symbolic expressions of Lyapunov quantities, and the proof of correctness of the suggested method are due to the author.

The content of the third Chapter is the application of qualitative theory to differential equations for the study of mathematical models of phase synchronization such as the systems of connected pendulums in the Huygens problem and the control systems by frequency of phase-locked loops.

In 1669 Christian Huygens discovered that two pendulum clocks attached to a common support beam converged (regardless of the initial conditions), after

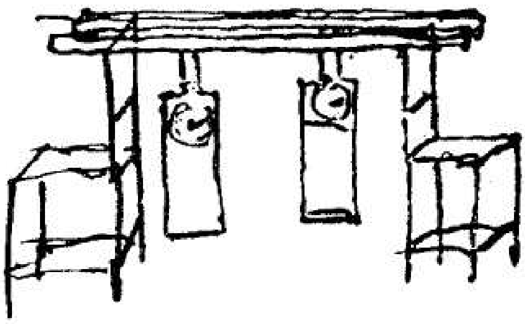

FIGURE 1 Drawing by Christian Huygens of two pendulum clocks attached to a beam which is supported by chairs

some transient, to an oscillatory regime with identical frequency of oscillations, while two pendulum angles moved in anti-phase.

The experiments carried out by the group of Professor H. Nijmeijer include a similar setup with two metronomes on a common support, and they demonstrate, along with anti-phase oscillations, an in-phase synchronization, where metronomes' pendulums agree not only in frequency but also in angles [Oud *et. al.,* 2006]. In Section 3.1 a mathematical model of system, consisting of two metronomes resting on a light wooden board that sits on two empty soda cans, is considered. The problem of analytical consideration of in-phase synchronization was also studied. This work was completed by the author together with Prof. Nijmeijer, Prof. Leonov, and Dr. A. Pogromsky within the framework of a Russian-Dutch research project [Kuznetsov *et al.,* 2007[1]].

In Section 3.2 the analysis of phase-looked loops operation is carried out.

A phase-locked loop (PLL) had been invented in the 1930s-1940s [De Bellescize, 1932; Wendt, Fredentall, 1943] and nowadays is frequently encountered in

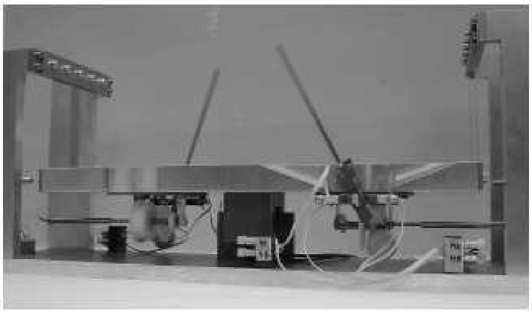

FIGURE 2 Experimental setup in Eindhoven Technical University

radio engineering and communication. One of the first applications of the phase-locked loop is related to the problems of data transfer by means of radio signal. In radio engineering the PLL is applied to a carrier synchronization, carrier recovery, demodulation, and frequency division and multiplication.

In the last ten years, PLLs have widely been used in array processors and other devices of digital information processing. The main requirement to PLLs for array processors is that they must be floating in phase. This means that the system must eliminate completely a clock skew. The elimination of clock skew is one of the most important problems in parallel computing and information processing (as well as in the design of array processors [Kung, 1988]). Several approaches to solving the problem of eliminating the clock skew have been devised for the last thirty years. When developed the design of multiprocessor systems, it was suggested [Kung, 1988] joining the processors in the form of an H-tree in which the lengths of the paths from the clock to every processor are the same. However, in this case the clock skew is not eliminated completely because of the

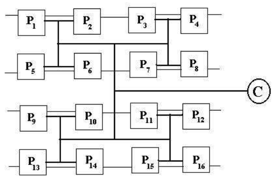

FIGURE 3 H-tree

heterogeneity of wires [Kung, 1988]. Moreover, for a great number of processors, the configuration of communication wires is very complicated. This leads to various difficult technological problems.

The increase in the number of processors in multiprocessor systems required an increase in the power of the clock. But the powerful clock came to produce significant electromagnetic noise. About ten years ago, a new approach to eliminating the clock skew and reducing generator's power was put forward. It consists of introducing a special distributed system of clocks controlled by PLL. This approach enables us to reduce significantly the power of clocks.

Also, in the present work the classical ideas by Viterbi [Viterbi, 1966] are

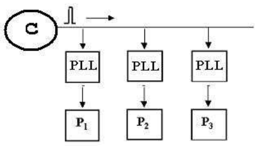

FIGURE 4 Distributed system of clocks controlled by PLLs

extended and generalized for the design of PLL with pulse modulation. Using rigorous mathematical analysis, new methods for the design of different block diagrams of PLL are proposed. All results presented below were reported at different international conferences [Kuznetsov *et al.*, 2006[2]; Kuznetsov *et al.*, 2007[2]; Kuznetsov *et al.*, 2008[1]].

1 STABILITY BY THE FIRST APPROXIMATION[1]

1.1 Introduction

In 1892, the general problem of stability by the first approximation was considered by Lyapunov [Lyapunov, 1892]. He proved that if the system of the first approximation is regular and all its characteristic exponents are negative, then the solution of the original system is asymptotically Lyapunov stable. In 1930, it was stated by O. Perron [Perron, 1930] that the requirement of regularity of the first approximation is substantial . He constructed an example of the second-order system of the first approximation, which has negative characteristic exponents along a zero solution of the original system but, at the same time, this zero solution of original system is Lyapunov unstable. Furthermore, in a certain neighborhood of this zero solution almost all solutions of original system have positive characteristic exponents. The effect of a sign reversal of characteristic exponents of solutions of the original system and the system of first approximation with the same initial data was subsequently called the Perron effect [Leonov, 1998; 2002[1];2002[2];2003;2004;2008]. The counterexample of Perron impressed on the contemporaries and gave an idea of the difficulties arising in the justification of the first approximation theory for nonautonomous and nonperiodic linearizations. Later, Persidskii [1947], Massera [1957], Malkin [1966], and Chetaev [1955], obtained sufficient conditions of stability by the first approximation for nonregular linearizations generalizing the Lyapunov theorem.

At the same time, according to Malkin ([1966], pp. 369–370): ([translated from Russian into English)

"... The counterexample of Perron shows that the negativeness of characteristic exponents is not a sufficient condition of stability by the first approximation. In the general case necessary and sufficient conditions of stability by the first approximation are not obtained."

---

[1] This chapter is mostly based on the survey *Leonov G.A., Kuznetsov N.V. "Time-Varying Linearization and the Perron effects", Int. J. of Bifurcation and Chaos, Vol. 17, No. 4, 2007, pp. 1079–1107*. Reprinted with kind permission of World Scientific Publishing Co.

For certain situations, the results on stability by the first approximation can be found in the books [Bellman & Cooke, 1963, Davies & James, 1966, Willems, 1970, Coppel, 1965, Wasow, 1965, Bellman, 1953, Bacciotti & Rosier, 2005, Yoshizawa, 1966, Lakshmikantham *et al.,* 1989, Hartman, 1984, Halanay, 1966, Sansone & Conti 1964].

In the 1940s Chetaev [1948] published the criterion of instability by the first approximation for regular linearizations. However, in the proof of these criteria a flaw was discovered [Leonov, 2008, Leonov, 2004] and, at present, the complete proof of Chetaev theorems is given for a more weak condition in comparison with that for instability by Lyapunov, namely, for instability by Krasovsky only [Leonov, 2002^1].

The discovery of strange attractors was made obvious with the study of instability by the first approximation.

Nowadays the problem of the justification of the nonstationary linearizations for complicated nonperiodic motions on strange attractors bears a striking resemblance to the situation that occurred 120 years ago. The founders of the automatic control theory D.K. Maxwell [1868], and A.I. Vyschegradskii [1877] courageously used a linearization in a neighborhood of stationary motions, leaving the justification of such linearization to A. Poincare [1886] and A.M. Lyapunov [1892]. At present, many specialists in chaotic dynamics believe that the positiveness of the largest characteristic exponent of a linear system of the first approximation implies the instability of solutions of the original system (see, for example [Schuster, 1984, Moon, 1987, Neimark & Landa, 1992, Heagy & Hammel, 1994, Wu *et al.*, 1994, Ryabov, 2002]

Moreover, there is a number of computer experiments, in which the various numerical methods for calculating the characteristic exponents and the Lyapunov exponents of linear systems of the first approximation are used [Luca & Vleck, 2002, Luca *et al.,* 1997, Goldhirsch *et al.*, 1987].

As a rule, authors ignore the justification of the linearization procedure and use the numerical values of exponents so obtained to construct various numerical characteristics of attractors of the original nonlinear systems (Lyapunov dimensions, metric entropies, and so on). Sometimes, computer experiments serve as arguments for the partial justification of the linearization procedure. For example, some computer experiments [Russel *et al.,* 1980, Neimark & Landa, 1992] show the coincidence of the Lyapunov and Hausdorff dimensions of the attractors of Henon, Kaplan–Yorke and Zaslavskii. But for B-attractors of Henon and Lorenz, such a coincidence does not hold [Leonov, 2001^2, Leonov, 2002^2].

So, the approach based on linearizations along the trajectories on the strange attractors require justification. This motivates to the development of the nonstationary theory of instability by the first approximation.

This chapter shows the contemporary state of the art of the problem of the justification of nonstationary linearizations. Here for the discrete and continuous systems the results of stability by the first approximation for regular and nonregular linearizations are given, the Perron effects are considered, the criteria of the stability and instability of the flow and cascade of solutions and the criteria of instability by Lyapunov and Krasovsky are obtained.

1.2 Classical Definitions of Stability

Consider a continuous system

$$\frac{dx}{dt} = F(x,t), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{1}$$

where the vector-function $F(\cdot,\cdot)\colon\ \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous.

Along with system (1) we consider its discrete analog [LaSalle, 1980, Bromberg, 1986, Leonov, 2001]

$$x(t+1) = F\big(x(t),t\big), \qquad x \in \mathbb{R}^n,\ t \in \mathbb{Z}, \tag{1'}$$

where $F(\cdot,\cdot)\colon\ \mathbb{R}^n \times \mathbb{Z} \to \mathbb{R}^n$.

For Eq. (1) there are many theorems of existence and uniqueness [Cesari, 1959, Filippov, 1988, Hartman, 1984]. For (1′) it is readily shown that in all cases the trajectory is defined for all $t = 0, 1, 2, \ldots$.

We consider the solution $x(t)$ of system (1) or (1′), given on the interval $a < t < +\infty$.

Definition 1 *The solution $x(t)$ is said to be Lyapunov stable if for any $\varepsilon > 0$ and $t_0 > a$ there exists a number $\delta = \delta(\varepsilon, t_0)$ such that*

1) *all solutions $y(t)$, satisfying the condition*

$$|y(t_0) - x(t_0)| < \delta,$$

are defined in the interval $t_0 \le t < +\infty$;

2) *for these solutions the inequality*

$$|x(t) - y(t)| < \varepsilon, \quad \forall t \ge t_0$$

is valid.

If $\delta(\varepsilon, t_0)$ is independent of t_0, then the Lyapunov stability is called uniform.

Definition 2 *The solution $x(t)$ is said to be asymptotically Lyapunov stable if it is Lyapunov stable and for any $t_0 > a$ there exists a positive number $\Delta = \Delta(t_0)$ such that all solutions $y(t)$, defined in the interval $t_0 \le t < +\infty$ and satisfying the condition $|y(t_0) - x(t_0)| < \Delta$, have the following property:*

$$\lim_{t\to+\infty} |y(t) - x(t)| = 0.$$

In other words, for any $\varepsilon' > 0$ there exists a positive number $T = T(\varepsilon', y(t_0), t_0)$ such that the inequality

$$|x(t) - y(t)| < \varepsilon', \quad \forall t \ge t_0 + T$$

is valid.

If $x(t)$ is uniformly stable and $\Delta(t_0)$ and $T(\varepsilon', y(t_0), t_0)$ is independent of t_0, then the Lyapunov asymptotic stability is called uniform.

Definition 3 *The solution $x(t)$ is said to be exponentially stable if for any $t_0 > a$ there exist positive numbers $\delta = \delta(t_0)$, $R = R(t_0)$, and $\alpha = \alpha(t_0)$ such that*

1) *all solutions $y(t)$, satisfying the condition*

$$|y(t_0) - x(t_0)| < \delta,$$

are defined in the interval $t_0 \le t < +\infty$;

2) *the inequality*

$$|y(t) - x(t)| \le R \exp\big(-\alpha\,(t - t_0)\big)|y(t_0) - x(t_0)|, \quad \forall t \ge t_0$$

is satisfied. If δ, R, and α are independent of t_0, then the exponential stability is called uniform.

Assuming $\alpha = 0$, we obtain the definition of stability by Krasovsky [Leonov, 2002[1]].

Definition 4 *The solution $x(t)$ is said to be Krasovsky stable if for any $t_0 > a$ there exist positive numbers $\delta = \delta(t_0)$ and $R = R(t_0)$ such that*

1) *all solutions $y(t)$, satisfying the condition*

$$|y(t_0) - x(t_0)| < \delta,$$

are defined in the interval $t_0 \le t < +\infty$;

2) *the following inequality*

$$|x(t) - y(t)| \le R|y(t_0) - x(t_0)|, \quad \forall t \ge t_0$$

is valid. If δ and R are independent of t_0, then stability by Krasovsky is called uniform.

Hence, it follows that the stability of solution by Krasovsky yields its stability by Lyapunov. Relations with uniform stability can be found in [Willems, 1970].

If for any interval $(t_1, t_2) \subset (a, +\infty)$ the solutions, given on it, are continuable on the whole interval $(a, +\infty)$, then the following assertion is valid [Demidovich, 1967].

Suppose that the solution $x = x(t)$, given on the interval $a < t < +\infty$, is stable for the fixed moment $t_0 \in (a, +\infty)$. Then it is stable for any moment $t_0' \in (a, +\infty)$. Therefore, we can restrict ourselves by checking of the stability of solution for the certain given initial moment t_0 only.

Further the initial moment t_0 is assumed to be fixed.

Without loss of generality, we consider solutions with $t_0 = 0$. Denote by $x(t, x_0)$ a solution of either system (1) or system (1′) with the initial data:

$$x(0, x_0) = x_0,$$

and suppose that all solutions $x(t, x_0)$ of continuous system are defined on the interval $[0, +\infty)$ and the solutions of discrete system are defined on the set $\mathbb{N}_0 = 0, 1, 2\ldots$.

Consider the function $V(t,x)$ $(V(\cdot,\cdot)\colon [0,+\infty)\times \mathbb{R}^n \to \mathbb{R}^1)$ differentiable in a certain neighborhood of the point $x=0$. Introduce the following notation. Let

$$V(t,x)^\bullet = \frac{\partial}{\partial t}V(t,x) + \sum_i \frac{\partial V}{\partial x_i} F_i(t,x).$$

The function $V(t,x)^\bullet$ is often called a derivative of the function $V(t,x)$ along the solutions of system (1). Here x_i is the ith component of the vector x and F_i is the ith component of the vector-function F.

For the discrete system, consider the function $V(t,x)$ $(V(\cdot,\cdot)\colon \mathbb{N}_0\times \mathbb{R}^n \to \mathbb{R}^1)$, and let

$$\blacktriangle V\big(t,x\big) = V\big(t+1, x(t+1)\big) - V\big(t, x(t)\big),$$

where $x(t)$ is a solution of system (1′).

1.2.1 Reduction of the problem to the study of zero solution

The first procedure in studying the stability (or instability) of the solution $x(t,x_0)$ of systems (1) and (1′) is, as a rule, the following transformation

$$x = y + x(t,x_0).$$

Using this change of variable for continuous system (1) and discrete system (1′), we have, respectively, the equations

$$\frac{dy}{dt} = F\big(y + x(t,x_0), t\big) - F\big(x(t,x_0), t\big) \tag{2}$$

$$y(t+1) = F\big(y(t) + x(t,x_0), t\big) - F\big(x(t,x_0), t\big) \tag{2′}$$

which are often called equations of perturbed motion. It is evident that the problem of the stability of the solution $x(t,x_0)$ is reduced to the problem of the stability of the trivial solution $y(t)\equiv 0$.

In this case we assume that the right-hand sides of (2) and (2′) are known since $F(x,t)$ and the solution $x(t,x_0)$ are known. At present, the difficulties of the calculation of $x(t,x_0)$ can often be overcome with the help of numerical methods and computational experiments.

1.3 Characteristic Exponents, Regular Systems, Lyapunov Exponents

Consider systems (2) and (2′) with a marked linear part. In the continuous case we have

$$\frac{dx}{dt} = A(t)x + f(t,x),\ x\in\mathbb{R}^n,\quad t\in[0,+\infty), \tag{3}$$

where $A(t)$ is a continuous $(n\times n)$-matrix, $f(\cdot,\cdot) : [0,+\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector-function. Suppose,

$$\sup_{t\in[0,+\infty)} |A(t)| < +\infty$$

in the continuous case.

In the discrete case, we have

$$x(t+1) = A(t)x(t) + f\big(t, x(t)\big),\ x(t) \in \mathbb{R}^n, \quad t \in \mathbb{N}_0, \tag{3'}$$

where $A(t)$ is an $(n \times n)$-matrix, $f(\cdot,\cdot)\colon\ \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^n$. In this case, we assume that

$$\det A(t) \neq 0, \quad \sup_{t \in \mathbb{N}_0} |A(t)| < +\infty, \quad \sup_{t \in \mathbb{N}_0} |A(t)^{-1}| < +\infty.$$

Here $|\cdot|$ is the Euclidean norm.

Suppose, in a certain neighborhood $\Omega(0)$ of the point $x = 0$ the nonlinear parts of systems (3) and (3′) satisfy the following condition

$$|f(t,x)| \leq \kappa |x|^{\nu} \ \ \forall t \geq 0, \quad \forall x \in \Omega(0), \quad \kappa > 0, \nu > 1. \tag{4}$$

We shall say that the first approximation system for (3) *is the following linear system*

$$\frac{dx}{dt} = A(t)x \tag{5}$$

and that for discrete system (3′) *is the linear system*

$$x(t+1) = A(t)x(t). \tag{5'}$$

Consider a fundamental matrix $X(t) = \big(x_1(t), ..., x_n(t)\big)$, consisting of the linear-independent solutions $\{x_i(t)\}_1^n$ of the first approximation system. For the determinant of the fundamental matrix we have the Ostrogradsky–Liouville formula, which in the continuous case is as follows

$$\det X(t) = \det X(0) \exp\left(\int_0^t \operatorname{Tr} A(\tau) d\tau\right), \tag{6}$$

and in the discrete one takes the form

$$\det X(t) = \det X(0) \prod_{j=0}^{t-1} \det A(j). \tag{6'}$$

The fundamental matrices are often considered to satisfy the following condition

$$X(0) = I_n,$$

where I_n is a unit $(n \times n)$-matrix.

These definitions and results are valid for continuous system as well as for the discrete one. The proofs will be given, if necessary, for each situation separately.

Consider the vector-function $f(t)$ such that $\lim\limits_{t \to +\infty} \sup |f(t)| \neq 0$.

Definition 5 *The value (or the symbol $+\infty$, or $-\infty$), defined by formula*

$$\mathcal{X}[f(t)] = \lim_{t\to+\infty} \sup \frac{1}{t} \ln |f(t)|,$$

is called a characteristic exponent of the vector-function $f(t)$.

The characteristic exponent is equal to that taken with inverse sign characteristic number, introduced by Lyapunov [1892].

Definition 6 *The characteristic exponent of the vector-function $f(t)$ is said to be exact if the finite limit*

$$\mathcal{X}[f(t)] = \lim_{t\to+\infty} \frac{1}{t} \ln |f(t)|$$

exists.

Consider the characteristic exponents of solutions of linear system (5) or (5′).

Definition 7 *[Demidovich, 1967] A set of distinctive characteristic exponents of all solutions (except a zero solution), being different from $\pm\infty$, of linear system is called its spectrum.*

Note that the number of different characteristic exponents is bounded by the dimension of the considered space of system states.

1.3.1 Regular systems

Consider the normal fundamental systems of solutions introduced by Lyapunov [1892].

Definition 8 *A fundamental matrix is said to be normal if the sum of characteristic exponents of its columns is minimal in comparison with other fundamental matrices.*

For discrete [Demidovich, 1969, Gayschun, 2001] and continuous systems [Demidovich, 1967] the following result is well known.

Lemma 1

1) In all normal fundamental systems of solutions, the number of solutions with equal characteristic exponents is the same.

2) Each normal fundamental system realizes a spectrum of linear system.

Thus, we can introduce the following definition.

Definition 9 *[Demidovich, 1967] The set of characteristic exponents*

$$\lambda_1, ..., \lambda_n$$

of a certain normal fundamental system of solutions is called a complete spectrum and the number $\sigma = \sum_1^n \lambda_i$ is a sum of characteristic exponents of linear system.

Note that any fundamental system of solutions has a solution with the largest characteristic exponent $\max\limits_{1\le j\le n} \lambda_j$.

Consider a class of regular systems, introduced by Lyapunov.

Definition 10 *A linear system is said to be regular if for the sum of its characteristic exponents σ the following relation holds*

$$\sigma = \lim_{t\to+\infty} \inf \frac{1}{t} \ln|\det X(t)|.$$

Taking into account formula (6), in the continuous case we obtain a classical definition [Demidovich, 1967, Adrianova, 1998] of the regularity of system

$$\sigma = \lim_{t\to+\infty} \inf \frac{1}{t} \int\limits_0^t \operatorname{Tr} A(\tau)\, d\tau.$$

Similarly, formula (6′) gives a definition of regularity [Demidovich, 1969] in the discrete case

$$\sigma = \lim_{t\to+\infty} \inf \frac{1}{t} \ln \prod_{j=0}^{t-1} |\det A(j)|.$$

Definition 11 *The number*

$$\Gamma = \sigma - \lim_{t\to+\infty} \inf \frac{1}{t} \ln|\det X(t)|$$

is called an irregularity coefficient of linear system.

As was shown in [Demidovich, 1967], the systems with constant and periodic coefficients are regular.

For continuous [Demidovich, 1967] and discrete systems [Demidovich, 1969, Gayschun, 2001] the following is well known

Lemma 2 *(Lyapunov inequality)*

Let all characteristic exponents of solutions of linear system be $< +\infty$ (or all characteristic exponents be $> -\infty$.)

Then, for any fundamental system of solutions $X(t)$ the following inequality

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln|\det X(t)| \le \sigma_X, \tag{7}$$

where σ_X is a sum of characteristic exponents of the system of solutions $X(t)$, is satisfied.

Thus, for regular systems there exists the limit

$$\lim_{t\to+\infty} \frac{1}{t} \ln|\det X(t)|.$$

Note that from the condition of regularity of linear system it follows [Demidovich, 1967] that for its solutions $x(t) \ne 0$ there exist the limits

$$\lim_{t\to+\infty} \frac{1}{t} \ln|x(t)|.$$

Note 1 *As was shown in [Bylov et al., 1966], the opposite, generally speaking, is not valid. We give an example of nonregular system, all characteristic exponents of which are exact [Bylov et al., 1966].*

Consider system (5) with the matrix

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & (\cos \ln t - \sin \ln t - 1) \end{pmatrix}, \quad t \geq 1 \tag{8}$$

and its fundamental matrix $X(t)$

$$X(t) = \big(x_1(t), x_2(t)\big) = \begin{pmatrix} 1 & \int\limits_1^t e^{\gamma(\tau)}\, d\tau \\ 0 & e^{\gamma(t)} \end{pmatrix},$$

where $\gamma(t) = t(\cos \ln t - 1)$. In this case for the determinant of fundamental matrix the following relation

$$\lim_{t\to+\infty} \inf \frac{1}{t} \ln |\det X(t)| = -2 \tag{9}$$

is satisfied. We now find characteristic exponents of solutions. For $x_1(t)$ we have

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln |x_1(t)| = \lim_{t\to+\infty} \inf \frac{1}{t} \ln |x_1(t)| = 0. \tag{10}$$

Since $e^{\gamma(t)} \leq 1$ for $t \geq 1$, we conclude that the characteristic exponent $x_2(t)$ is less than or equal to zero

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln |x_2(t)| \leq 0.$$

On the other hand, since the integral of $e^{\gamma(\tau)}$ is divergent, namely

$$\int\limits_1^{+\infty} e^{\gamma(\tau)}\, d\tau = +\infty, \tag{11}$$

for $x_2(t)$ we have the following estimate

$$\lim_{t\to+\infty} \inf \frac{1}{t} \ln |x_2(t)| \geq 0.$$

This implies that

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln |x_2(t)| = \lim_{t\to+\infty} \inf \frac{1}{t} \ln |x_2(t)| = 0. \tag{12}$$

Thus, by (9), (10), and (12) the linear system with matrix (8) has exact characteristic exponents but it is nonregular:

$$\Gamma = 2.$$

We prove that the integral of $e^{\gamma(\tau)}$ is divergent.

Suppose, $t^u(k) = e^{2k\pi+\delta(k)}$ and $t^l(k) = e^{2k\pi-\delta(k)}$, where $\delta(k) = e^{-k\pi}$, $k = 1, 2\ldots$. From the definition of $t^l(k)$ and $t^u(k)$ we obtain

$$t^u(k) - t^l(k) \geq e^{2k\pi-\delta(k)}(e^{2\delta(k)} - 1) \geq e^{2k\pi-\delta(k)} 2\delta(k) \geq 2e^{k\pi-1}. \qquad (13)$$

In the case $\tau \in \left[t^l(k), t^u(k)\right]$ for $\gamma(\tau)$ the estimate

$$-\gamma(\tau) \leq \tau\big(1 - \cos(\delta(k))\big) \leq t^u(k)\frac{\delta^2(k)}{2} \leq \frac{1}{2}e^{2k\pi+\delta(k)}e^{-2\pi k} \leq \frac{e^{\delta(k)}}{2} \leq \frac{e}{2} \qquad (14)$$

is valid. Then we have

$$\int\limits_1^{t^u(k)} e^{\gamma(\tau)} \mathrm{d}\tau \geq \left(t^u(k) - t^l(k)\right) e^{-e/2} \geq 2e^{k\pi-1-e/2} \to +\infty$$

as $k \to +\infty$.

■

1.3.2 Boundedness conditions for characteristic exponents

Consider the boundedness conditions for characteristic exponents of linear systems (5) and (5′).

We shall show that the boundedness of the norm of matrix $|A(t)|$ of linear continuous system (5) implies that the characteristic exponents are finite. This result follows from the inequality of Vazhevsky [Demidovich, 1967] for continuous system. We shall formulate it in the required form.

Theorem 1 *For solutions of system* (5) *the inequalities*

$$|x(\tau)| \exp \int\limits_\tau^t \alpha(s)\, ds \leq |x(t)| \leq |x(\tau)| \exp \int\limits_\tau^t \beta(s)\, ds, \quad \forall t \geq \tau,$$
$$|x(\tau)| \exp \int\limits_\tau^t \beta(s)\, ds \leq |x(t)| \leq |x(\tau)| \exp \int\limits_\tau^t \alpha(s)\, ds, \quad \forall t \leq \tau$$

are satisfied.

Here $\alpha(t)$ and $\beta(t)$ are the smallest and largest eigenvalues, respectively, of the matrix

$$\frac{1}{2}[A(t) + A(t)^*].$$

Proof.
Since the inequalities

$$\alpha(t)|x|^2 \leq \frac{1}{2}x^*[A(t) + A(t)^*]x \leq \beta(t)|x|^2, \quad \forall x \in \mathbb{R}^n$$

and the relation

$$\frac{d}{dt}(|x(t)|^2) = x(t)^*[A(t) + A(t)^*]x(t),$$

are satisfied, we have the following estimate

$$2\alpha(t) \leq \frac{(|x(t)|^2)^{\bullet}}{|x(t)|^2} \leq 2\beta(t).$$

Integrating these inequalities from τ to t, for $t \geq \tau$ we obtain the first estimate in the theorem and for $t \leq \tau$ the second one. ■

Corollary 1 *If the norm of the matrix $A(t)$ of linear continuous system is bounded on $\mathbb{R}^1$, then there exists a number ν such that for any t and τ the estimate*

$$|X(t)X(\tau)^{-1}| \leq \exp\left(\nu|t-\tau|\right) \tag{15}$$

is valid.

Estimate (15) results from Theorem 1 and from the following obvious relation

$$X(t)X(\tau)^{-1}x(\tau) = x(t).$$

Actually, we can choose ν in such a way that the following inequalities

$$|X(t)X(\tau)^{-1}| = \max_{y\neq 0} \frac{|X(t)X(\tau)^{-1}y|}{|y|} \leq$$

$$\leq \max\left(\exp\int_{\tau}^{t}\beta(s)\,ds,\ \exp\int_{\tau}^{t}\alpha(s)\,ds\right) \leq \exp\left(\nu|t-\tau|\right)$$

are satisfied. ■

Consider now the discrete case.

Lemma 3 *[Demidovich, 1969] If the conditions*

$$1.\quad \sup_{t\in\mathbb{N}_0}|A(t)| < +\infty,$$
$$2.\quad \sup_{t\in\mathbb{N}_0}|A(t)^{-1}| < +\infty$$

are satisfied, then each nontrivial solution of system (5′) *has a finite characteristic exponent.*

Proof of lemma. Consider together with system (5′) its conjugate system

$$z(t+1) = A(t)^{-1^*}z(t), \quad z(t) \in \mathbb{R}^n, \quad t = 0, 1, \dots,$$

and its fundamental matrix

$$Z(t) = \prod_{j=1}^{t} A(t-j)^{-1^*} = A(t-1)^{-1^*}A(t-2)^{-1^*}\dots A(0)^{-1^*}.$$

Here A^* is Hermitian conjugate matrix.

For any solution $x(t)$ of system ($5'$) we have the following estimate

$$|x(t)| \leq |x(0)| \left| \prod_{j=1}^{t} A(t-j) \right|.$$

Then by condition 1 of lemma, for all nonzero solutions of system ($5'$) we obtain

$$\mathcal{X}[x(t)] < +\infty. \tag{16}$$

In this case, Lyapunov inequality (7) implies that

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln \left| \det X(t) \right| < +\infty.$$

We have a similar inequality also for the conjugate system

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln \left| \left(\det X(t) \right)^{-1} \right| = \lim_{t\to+\infty} \sup \frac{1}{t} \ln \left| \det Z(t) \right| < +\infty.$$

Then

$$\lim_{t\to+\infty} \inf \frac{1}{t} \ln |\det X(t)| > -\infty.$$

By the Lyapunov inequality, we have

$$\sigma_X \geq \lim_{t\to+\infty} \sup \frac{1}{t} \ln \left| \det X(t) \right| \geq \lim_{t\to+\infty} \inf \frac{1}{t} \ln \left| \det X(t) \right| > -\infty,$$

where σ_X is a sum of characteristic exponents of solutions of the fundamental system $X(t)$. By (16) we have

$$-\infty < \mathcal{X}[x(t)] < +\infty.$$

■

Further extension of this result can be found in [Kuznetsov & Leonov, 2005[2]]:

Preposition 1 *If for the matrix of linear system* ($5'$) *the following inequalities*

$$1. \quad \lim_{t\to+\infty} \sup \frac{1}{t} \ln \left| \prod_{j=1}^{t} A(t-j) \right| < +\infty;$$

$$2. \quad \lim_{t\to+\infty} \sup \frac{1}{t} \ln \left| \prod_{j=1}^{t} A(t-j)^{-1} \right| < +\infty,$$

are satisfied, then each nontrivial solution of system ($5'$) *has a finite characteristic exponent.*

1.3.3 Lyapunov exponents and singular values

Consider singular values of the matrix $X(t)$ [Hahn, 1967, Katok & Hasselblat, 1995, Boichenko *et al.*, 2005].

Definition 12 *The singular values* $\{\alpha_j(X(t))\}_1^n$ *of the matrix* $X(t)$ *are the square roots of the eigenvalues of the matrix* $X(t)^*X(t)$.

The following geometric interpretation of singular values is known: the numbers $\alpha_j(X(t))$ coincide with a principal semiaxis of the ellipsoid $X(t)B$, where B is a ball of unit radius.

Definition 13 *[Temam, 1988] The Lyapunov exponent* μ_j *is as follows*

$$\mu_j = \lim_{t\to+\infty} \sup \frac{1}{t} \ln \alpha_j(X(t)). \tag{17}$$

In the case (17) the term "upper singular exponent" is also used [Barabanov, 2005].

Let μ_1 and λ_1 be the largest Lyapunov exponent and the largest characteristic exponent, respectively.

Lemma 4 *For the linear systems the largest characteristic exponent is equal to the largest Lyapunov exponent.*

Proof. Recall that a geometric interpretation of singular values implies the relation $|X(t)| = \alpha_1(X(t))$. Here $|X|$ is a norm of the matrix X, defined by formula

$$|X| = \max_{|x|=1} |Xx|, \quad x \in \mathbb{R}^n.$$

Then the relation

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln |X(t)| = \lambda_1$$

yields the relation $\lambda_1 = \mu_1$. ■

Note 2 *We will show that there exist systems such that the characteristic exponents do not coincide with the Lyapunov exponents*

We give an example [Leonov, 2008] of such system in the continuous case. Consider system (5) with the matrix

$$A(t) = \begin{pmatrix} 0 & \sin(\ln t) + \cos(\ln t) \\ \sin(\ln t) + \cos(\ln t) & 0 \end{pmatrix} \quad t > 1$$

and with the fundamental normal matrix

$$X(t) = \begin{pmatrix} e^{\gamma(t)} & e^{-\gamma(t)} \\ e^{\gamma(t)} & -e^{-\gamma(t)} \end{pmatrix},$$

where $\gamma(t) = t\sin(\ln t)$. It is obvious that $\lambda_1 = \lambda_2 = 1$ and

$$\alpha_1(X(t)) = \sqrt{2}\max(e^{\gamma(t)}, e^{-\gamma(t)})$$

$$\alpha_2(X(t)) = \sqrt{2}\min(e^{\gamma(t)}, e^{-\gamma(t)}).$$

This implies the following relations $\mu_1 = 1$, $\mu_2 = 0$. Thus, we have $\lambda_2 \neq \mu_2$.

Now we present a criterion of regularity for linear system in terms of Lyapunov exponents.

Lemma 5 *[Barabanov, 2005] A linear system is regular if and only if*

$$\lim_{t\to+\infty} \inf \frac{1}{t}\ln\alpha_j\big(X(t)\big) = \lim_{t\to+\infty} \sup \frac{1}{t}\ln\alpha_j\big(X(t)\big), \quad j = 1, .., n. \tag{18}$$

1.3.4 Estimates of a norm of Cauchy matrix

Consider a normal fundamental matrix $X(t)$ of linear system, and let

$$\Lambda = \max_j \lambda_j, \quad \lambda = \min_j \lambda_j.$$

Here $\{\lambda_j\}_1^n$ is a complete spectrum of linear system.

We shall say that $X(t)X(\tau)^{-1}$ is a Cauchy matrix.

The following result is well known and is often used.

Theorem 2 *[Leonov, 2008] For any number $\varepsilon > 0$ there exists a number $C > 0$ such that the following inequalities*

$$\begin{gathered} |X(t)X(\tau)^{-1}| \le C\exp\big((\Lambda+\varepsilon)(t-\tau) + (\Gamma+\varepsilon)\tau\big), \\ \forall\, t \ge \tau \ge 0 \end{gathered} \tag{19}$$

$$\begin{gathered} |X(t)X(\tau)^{-1}| \le C\exp\big(\lambda(t-\tau) + (\Gamma+\varepsilon)\tau\big), \\ \forall\, \tau \ge t \ge 0 \end{gathered}, \tag{20}$$

where Γ is the irregularity coefficient, are satisfied.

Proof. Let

$$X(t) = \big(x_1(t), \dots, x_n(t)\big), \quad \widetilde{x}_j(t) = x_j(t)\exp\big((-\lambda_j - \varepsilon)t\big)$$

$$X(t)^{-1} = \begin{pmatrix} u_1(t)^* \\ \vdots \\ u_n(t)^* \end{pmatrix}, \quad \widetilde{u}_j(t) = u_j(t)\exp\big((\lambda_j+\varepsilon)t\big),$$

$$\Sigma = \sum_1^n \lambda_j.$$

From the definition of λ_j and by the rule of matrix inversion we obtain that for a certain number $L > 0$ the inequality

$$\Big|\big(\widetilde{x}_1(t), \dots, \widetilde{x}_n(t)\big)^{-1}\det\big(\widetilde{x}_1(t), \dots, \widetilde{x}_n(t)\big)\Big| \le L, \ \forall\, t \ge 0 \tag{21}$$

is satisfied. A sufficiently large number L_1 exists such that relation (21) yields the estimate

$$\big|\big(\widetilde{x}_1(t),\ldots,\widetilde{x}_n(t)\big)^{-1}\big| \leq L\exp\big((\Sigma+n\varepsilon)t-\ln\big|\det X(t)\big|\big) \leq \\ \leq L_1\exp\big((2n\varepsilon+\Gamma)t\big),\ \forall t\geq 0. \tag{22}$$

We have the following obvious relations:

$$|X(t)X(\tau)^{-1}| = |\sum_j x_j(t)u_j(\tau)^*| = \\ = \big|\sum_j \widetilde{x}_j(t)\exp\big((\lambda_j+\varepsilon)t-(\lambda_j+\varepsilon)\tau\big)\widetilde{u}_j(\tau)^*\big|.$$

Taking into account that for $t \geq 0$ the vector-function $\widetilde{x}_j(t)$ is bounded, for sufficiently large number L_2 we obtain the following estimate

$$\big|X(t)X(\tau)^{-1}\big| \leq L_2\sum_j \exp\big((\lambda_j+\varepsilon)(t-\tau)\big)|\widetilde{u}_j(\tau)|. \tag{23}$$

Since

$$\begin{pmatrix}\widetilde{u}_1(t)^*\\ \vdots\\ \widetilde{u}_n(t)^*\end{pmatrix} = \big(\widetilde{x}_1(t),\ldots,\widetilde{x}_n(t)\big)^{-1},$$

by (22) and (23) we obtain the estimates (19) and (20). ■

1.3.5 The Nemytskii — Vinograd counterexample

Consider a continuous system [Bylov *et al.*, 1966]

$$\frac{dx}{dt} = A(t)x$$

with the matrix

$$A(t) = \begin{pmatrix}1-4(\cos 2t)^2 & 2+2\sin 4t\\ -2+2\sin 4t & 1-4(\sin 2t)^2\end{pmatrix}.$$

In this case, its solution is the vector-function

$$x(t) = \begin{pmatrix}e^t\sin 2t\\ e^t\cos 2t\end{pmatrix}. \tag{24}$$

It follows that

$$\det(A(t)-pI_n) = p^2+2p+1.$$

Therefore for the eigenvalues $\nu_1(t)$ and $\nu_2(t)$ of the matrix $A(t)$ we have

$$\nu_1(t) = \nu_2(t) = -1.$$

On the other hand, the characteristic exponent λ of solution (24) is equal to 1.

This counterexample shows that all eigenvalues of the matrix $A(t)$ can have negative real parts even if the corresponding linear system has positive characteristic exponents.

It also shows that the formulas, obtained in the book [Anishchenko *et al.*, 2002], namely

$$\lambda_j = \limsup_{t\to+\infty} \frac{1}{t}\int_0^t \mathrm{Re}\nu_j(\tau)\,d\tau$$

are untrue.

1.4 The Perron Effects

In 1930, O. Perron [Perron, 1930] showed that the negativeness of the largest characteristic exponent of the first approximation system does not always result in the stability of zero solution of the original system. Furthermore, in an arbitrary small neighborhood of zero, the solutions of the original system with positive characteristic exponent can be found. These results of Perron impressed on the specialists in the theory of stability of motion.

The effect of sign reversal of characteristic exponent of the solutions of the first approximation system and of the original system with the same initial data we shall call [Leonov, 1998, Leonov, 2004] the Perron effect.

We now present the outstanding result of Perron [1930] and its discrete analog [Kuznetsov & Leonov, 2001, Gayschun, 2001].

Consider the following system

$$\begin{aligned} \frac{dx_1}{dt} &= -ax_1 \\ \frac{dx_2}{dt} &= \big(\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a\big)x_2 + x_1^2 \end{aligned} \tag{25}$$

and its discrete analog

$$\begin{aligned} x_1(t+1) &= \exp(-a)x_1(t) \\ x_2(t+1) &= \frac{\exp\big((t+2)\sin\ln(t+2) - 2a(t+1)\big)}{\exp\big((t+1)\sin\ln(t+1) - 2at\big)}x_2(t) + x_1(t)^2. \end{aligned} \tag{25$'$}$$

Here a is a number satisfying the following inequalities

$$1 < 2a < 1 + \frac{1}{2}\exp(-\pi). \tag{26}$$

The solution of the first approximation system for systems (25) and (25′) takes the form

$$\begin{aligned} x_1(t) &= \exp(-at)x_1(0) \\ x_2(t) &= \exp\big((t+1)\sin(\ln(t+1)) - 2at\big)x_2(0). \end{aligned}$$

It is obvious that by condition (26) for the solution of the first approximation system for $x_1(0) \neq 0, x_2(0) \neq 0$ we have

$$\mathcal{X}[x_1(t)] = -a, \ \mathcal{X}[x_2(t)] = 1 - 2a < 0.$$

This implies that a zero solution of linear system of the first approximation is Lyapunov stable.

Now we consider the solution of system (25)

$$\begin{aligned} &x_1(t) = \exp(-at)x_1(0), \\ &x_2(t) = \exp\big((t+1)\sin(\ln(t+1)) - 2at\big)\times \\ &\times \left(x_2(0) + x_1(0)^2 \int_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)d\tau \right). \end{aligned} \tag{27}$$

Assuming $t = t_k = \exp\left((2k + \frac{1}{2})\pi\right) - 1$, where k is an integer, we obtain

$$\exp\big((t+1)\sin(\ln(t+1)) - 2at\big) = \exp\big((1-2a)t + 1\big), \quad (1+t)e^{-\pi} - 1 > 0,$$

$$\int_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)d\tau >$$

$$> \int_{f(k)}^{g(k)} \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)\, d\tau >$$

$$> \int_{f(k)}^{g(k)} \exp\left(\frac{1}{2}(\tau+1)\right) d\tau > \int_{f(k)}^{g(k)} \exp\left(\frac{1}{2}(\tau+1)\exp(-\pi)\right) d\tau =$$

$$= \exp\left(\frac{1}{2}(t+1)\exp(-\pi)\right)(t+1)\left(\exp\big(-\frac{2\pi}{3}\big) - \exp(-\pi)\right),$$

where

$$f(k) = (1+t)\exp(-\pi) - 1,$$

$$g(k) = (1+t)\exp\left(-\frac{2\pi}{3}\right) - 1.$$

Hence we have the following estimate

$$\exp\big((t+1)\sin(\ln(t+1)) - 2at\big)\int_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)d\tau >$$

$$\begin{aligned} &> \exp\left(\frac{1}{2}(2+\exp(-\pi))\right)\left(\exp\big(-\frac{2\pi}{3}\big) - \exp(-\pi)\right)(t+1)\times \\ &\times \exp\left(\big(1 - 2a + \frac{1}{2}\exp(-\pi)\big)t\right). \end{aligned} \tag{28}$$

From the last inequality and condition (26) it follows that for $x_1(0) \neq 0$ one of the characteristic exponents of solutions of system (25) is positive:

$$\mathcal{X}[x_1(t)] = -a,\ \mathcal{X}[x_2(t)] \geq 1 - 2a + e^{-\pi}/2 > 0. \tag{29}$$

Thus, we obtain that all characteristic exponents of the first approximation system are negative but almost all solutions of the original system (25) tend exponentially to infinity as $t_k \to +\infty$.

Consider now the solution of discrete system (25′)

$$\begin{aligned} x_1(t) &= x_1(0)e^{-at} \\ x_2(t) &= \exp\big((t+1)\sin\ln(t+1) - 2at\big)\times \\ &\times\Big(x_2(0) + x_1(0)^2\sum_{k=0}^{t-1}\exp\big(-(k+2)\sin\ln(k+2)+2a\big)\Big), \end{aligned} \tag{30}$$

and show that for this system inequalities (29) are also satisfied. For this purpose we obtain the estimate similar to estimate (28) in the discrete case.

Obviously, for any $N > 0$ and $\delta > 0$ there exists a natural number $(t' = t'(N,\delta),\ t' > N)$ such that

$$\sin\ln(t'+1) > 1 - \delta.$$

Then

$$\exp\big((t'+1)\sin\ln(t'+1) - 2at'\big) \geq \exp\big((1-\delta-2a)t' + 1 - \delta\big). \tag{31}$$

Now we estimate from below the second multiplier in the expression for $x_2(t)$. For sufficiently large t' there exists a natural number m

$$m \in \left(\frac{t'+1}{e^{\pi}} - 2, t'\right)$$

such that

$$\sin\ln(m+2) \leq -\frac{1}{2}.$$

Then we have

$$-(m+2)\sin\ln(m+2) + 2a \geq \frac{t'+1}{2e^{\pi}}.$$

This implies the following estimate

$$\sum_{k=0}^{t'-1}\exp\big(-(k+2)\sin\ln(k+2)+2a\big) \geq \exp\left((t'+1)\frac{1}{2}e^{-\pi}\right). \tag{32}$$

From (31), (32) and condition (26) it follows that for $x_1(0) \neq 0$ one of characteristic exponents of solutions (30) of system (25′) is positive and inequalities (29) are satisfied.

We give an example, which show the possibility of the sign reversal of characteristic exponents “on the contrary", namely the solution of the first approximation system has a positive characteristic exponent while the solution of the original system with the same initial data has a negative exponent [Leonov, 2003].

Consider the following continuous system [Leonov, 2004]

$$\begin{aligned} \dot{x}_1 &= -ax_1 \\ \dot{x}_2 &= -2ax_2 \\ \dot{x}_3 &= \big(\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a\big)x_3 + x_2 - x_1^2 \end{aligned} \tag{33}$$

and its discrete analog

$$\begin{aligned} x_1(t+1) &= e^{-a}x_1(t) \\ x_2(t+1) &= e^{-2a}x_2(t) \\ x_3(t+1) &= \frac{\exp\big((t+2)\sin\ln(t+2) - 2a(t+1)\big)}{\exp\big((t+1)\sin\ln(t+1) - 2at\big)}x_3(t) + x_2(t) - x_1(t)^2 \end{aligned} \tag{33$'$}$$

on the invariant manifold

$$M = \{x_3 \in \mathbb{R}^1,\ x_2 = x_1^2\}.$$

Here the value a satisfies condition (26).

The solutions of (33) and (33′) on the manifold M take the form

$$\begin{aligned} x_1(t) &= \exp\big(-at\big)x_1(0) \\ x_2(t) &= \exp\big(-2at\big)x_2(0) \\ x_3(t) &= \exp\big((t+1)\sin(\ln(t+1)) - 2at\big)x_3(0), \\ x_1(0)^2 &= x_2(0). \end{aligned} \tag{34}$$

Obviously, these solutions have negative characteristic exponents.

For system (33) in the neighborhood of its zero solution, consider the system of the first approximation

$$\begin{aligned} \dot{x}_1 &= -ax_1 \\ \dot{x}_2 &= -2ax_2 \\ \dot{x}_3 &= \big(\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a\big)x_3 + x_2. \end{aligned} \tag{35}$$

The solutions of this system are the following

$$\begin{aligned} x_1(t) &= \exp\big(-at\big)x_1(0) \\ x_2(t) &= \exp\big(-2at\big)x_2(0) \\ x_3(t) &= \exp\big((t+1)\sin(\ln(t+1)) - 2at\big)\times \\ &\times\Big(x_3(0) + x_2(0)\int_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)d\tau\Big). \end{aligned} \tag{36}$$

For system (33′) in the neighborhood of its zero solution, the system of the first approximation is as follows

$$\begin{aligned} x_1(t+1) &= \exp(-a)x_1(t) \\ x_2(t+1) &= \exp(-2a)x_2(t) \\ x_3(t+1) &= \frac{\exp\big((t+2)\sin\ln(t+2) - 2a(t+1)\big)}{\exp\big((t+1)\sin\ln(t+1) - 2at\big)}x_3(t) + x_2(t). \end{aligned} \tag{35$'$}$$

Then the solutions of system (35′) take the form

$$
\begin{aligned}
x_1(t) &= \exp\big(-at\big)x_1(0)\\
x_2(t) &= \exp\big(-2at\big)x_2(0)\\
x_3(t) &= \exp\big((t+1)\sin\ln(t+1)-2at\big)\times\\
&\times\Big(x_3(0)+x_2(0)^2\sum_{k=0}^{t-1}\exp\big(-(k+2)\sin\ln(k+2)+2a\big)\Big).
\end{aligned}
\tag{36′}
$$

By estimates (28) and (32) for solutions (36) and (36′) for $x_2(0) \neq 0$ we obtain

$$\mathcal{X}[x_3(t)] > 0.$$

It is easily shown that for solutions of systems (33) and (35) the following relations

$$\big(x_1(t)^2 - x_2(t)\big)^{\bullet} = -2a\big(x_1(t)^2 - x_2(t)\big)$$

are valid. Similarly, for system (35′) we have

$$x_1(t+1)^2 - x_2(t+1) = \exp(-2a)\big(x_1(t)^2 - x_2(t)\big).$$

Then

$$x_1(t)^2 - x_2(t) = \exp\big(-2at\big)\big(x_1(0)^2 - x_2(0)\big).$$

It follows that the manifold M is an invariant exponentially attractive manifold for solutions of continuous systems (33) and (35), and for solutions of discrete systems (33′) and (35′).

This means that the relation $x_1(0)^2 = x_2(0)$ yields the relation $x_1(t)^2 = x_2(t)$ for all $t \in \mathbb{R}^1$ and for any initial data we have

$$\big|x_1(t)^2 - x_2(t)\big| \leq \exp\big(-2at\big)\big|x_1(0)^2 - x_2(0)\big|.$$

Thus, systems (33) and (35) have the same invariant exponentially attractive manifold M on which almost all solutions of the first approximation system (35) have a positive characteristic exponent and all solutions of the original system (33) have negative characteristic exponents. The same result can be obtained for discrete systems (33′) and (35′).

The Perron effect occurs here on the whole manifold

$$\{x_3 \in \mathbb{R}^1,\ x_2 = x_1^2 \neq 0\}.$$

To construct the exponentially stable system, the first approximation of which has a positive characteristic exponent we represent system (33) in the following way

$$
\begin{aligned}
\dot{x}_1 &= F(x_1, x_2)\\
\dot{x}_2 &= G(x_1, x_2)\\
\dot{x}_3 &= \big(\sin\ln(t+1) + \cos\ln(t+1) - 2a\big)x_3 + x_2 - x_1^2.
\end{aligned}
\tag{37}
$$

Here the functions $F(x_1, x_2)$ and $G(x_1, x_2)$ have the form

$$F(x_1, x_2) = \pm 2x_2 - ax_1, \quad G(x_1, x_2) = \mp x_1 - \varphi(x_1, x_2),$$

in which case the upper sign is taken for $x_1 > 0$, $x_2 > x_1^2$ and for $x_1 < 0$, $x_2 < x_1^2$, the lower one for $x_1 > 0$, $x_2 < x_1^2$ and for $x_1 < 0$, $x_2 > x_1^2$.

The function $\varphi(x_1, x_2)$ is defined as

$$\varphi(x_1, x_2) = \begin{cases} 4ax_2 & \text{for} \quad |x_2| > 2x_1^2 \\ 2ax_2 & \text{for} \quad |x_2| < 2x_1^2. \end{cases}$$

The solutions of system (37) are regarded in the sense of A.F. Filippov [Filippov, 1988, Yakubovich *et al.*, 2004]. By definition of $\varphi(x_1, x_2)$ the following system

$$\begin{aligned} \dot{x}_1 &= F(x_1, x_2) \\ \dot{x}_2 &= G(x_1, x_2) \end{aligned} \tag{38}$$

on the lines of discontinuity $\{x_1 = 0\}$ and $\{x_2 = x_1^2\}$ has sliding solutions, which are given by the equations

$$x_1(t) \equiv 0, \quad \dot{x}_2(t) = -4ax_2(t)$$

and

$$\dot{x}_1(t) = -ax_1(t), \quad \dot{x}_2(t) = -2ax_2(t), \quad x_2(t) \equiv x_1(t)^2.$$

In this case the solutions of system (38) with the initial data $x_1(0) \neq 0$, $x_2(0) \in \mathbb{R}^1$ attain the curve $\{x_2 = x_1^2\}$ in a finite time, which is less than or equal to 2π.

This implies that for the solutions of system (37) with the initial data $x_1(0) \neq 0$, $x_2(0) \in \mathbb{R}^1$, $x_3(0) \in \mathbb{R}^1$, for $t \geq 2\pi$ we obtain the relations $F(x_1(t), x_2(t)) = -ax_1(t)$, $G(x_1(t), x_2(t)) = -2ax_2(t)$. Therefore, based on these solutions for $t \geq 2\pi$ system (35) is a system of the first approximation for system (37).

System (35), as was shown above, has a positive characteristic exponent. At the same time, all solutions of system (37) tend exponentially to zero. ∎

The considered technique permits us to construct the different classes of nonlinear continuous and discrete systems for which the Perron effects occur.

1.5 The Lyapunov Matrix Equation

In the continuous case the Lyapunov matrix equation has the form

$$\dot{H}(t) + P(t)^* H(t) + H(t)P(t) = -G(t) \tag{39}$$

with respect to the symmetric differentiable matrix $H(t)$ and in the discrete case it takes the form

$$P(t)^* H(t+1)P(t) - H(t) = -G(t) \tag{39'}$$

with respect to the symmetric matrix $H(t)$. Here $P(t)$ and $G(t)$ are the bounded $(n \times n)$-matrices (and, in addition, the continuous ones in (39)) $G^*(t) = G(t)$, $\forall t \geq 0$.

Denote by $X(t)$ the fundamental matrices of the continuous system

$$\frac{dx}{dt} = P(t)x$$

and of the discrete system

$$x(t+1) = P(t)x(t).$$

Let for certain constants $\alpha > 0$, $C > 0$, $\gamma \geq 0$ the following estimate

$$|X(s)X(t)^{-1}| \leq C \exp\big(-\alpha(s-t) + \gamma t\big), \quad \forall s \geq t \geq 0 \tag{40}$$

be valid. Then the solution of Eq. (39) is the matrix

$$H(t) = \int_t^{+\infty} \big(X(s)X(t)^{-1}\big)^* G(s) \big(X(s)X(t)^{-1}\big)\, ds \tag{41}$$

and the solution of Eq. (39′) is the matrix

$$H(t) = \sum_{s=t}^{\infty} \big(X(s)X(t)^{-1}\big)^* G(s) \big(X(s)X(t)^{-1}\big). \tag{41′}$$

This statement can be verified directly by means of substituting $H(t)$ in the original equation. The estimate of the type (40) is due to Theorem 2.

The convergence of relation for $H(t)$ results from estimate (40). In addition, from (40) it follows that there exists a number R such that

$$|H(t)| \leq R \exp\big(2\gamma t\big), \quad \forall\, t \geq 0. \tag{42}$$

Along with estimate (42) an important role is played by the lower estimate of the quadratic form $z^* H(t) z$, which can be found by using in the continuous and discrete cases the following

Theorem 3 *[Leonov, 2008] Suppose, estimate* (40) *is valid and the estimate*

$$z^* G(t) z \geq \delta |z|^2, \quad \forall t \geq 0,\ \forall z \in \mathbb{R}^n \tag{43}$$

holds for a certain positive number $\delta > 0$.

Then there exists a number $\varepsilon > 0$ such that the inequality

$$z^* H(t) z \geq \varepsilon |z|^2, \quad \forall\, t \geq 0, \quad \forall z \in \mathbb{R}^n \tag{44}$$

is satisfied.

Proof. Consider the continuous case. Condition (43) gives the following estimates

$$z^* H(t) z \geq \delta \int_t^{+\infty} \big|X(s)X(t)^{-1} z\big|^2 ds \geq$$

$$\geq \delta \int_t^{+\infty} \frac{|z|^2}{\big|(X(s)X(t)^{-1})^{-1}\big|^2} ds = \delta |z|^2 \int_t^{+\infty} \frac{1}{\big|X(t)X(s)^{-1}\big|^2} ds.$$

Then from the corollary of Theorem 1 we obtain

$$z^*H(t)z \geq \delta|z|^2 \int\limits_t^{+\infty} \exp\left(2\nu(t-s)\right) ds = \varepsilon|z|^2,$$

where

$$\varepsilon = \delta(2\nu)^{-1}.$$

In the continuous case, the theorem is proved.
Now we prove this theorem in the discrete case. Let

$$y_t(s) = X(s)X(t)^{-1}z.$$

By condition (43) we have $y_t(s)^*G(s)y_t(s) \geq 0$ for all $s \geq 0$. Then

$$z^*H(t)z = \sum_{s=t}^{\infty} y_t(s)^*G(s)y_t(s) = z^*G(t)z + \sum_{s=t+1}^{\infty} y_t(s)^*G(s)y_t(s) \geq \delta|z|^2.$$

■

Note that for $\gamma = 0$ the norm of the matrix $H(t)$ is bounded.
Consider now the first-order equation

$$\frac{dy}{dt} = q(t)y, \quad y \in \mathbb{R}, \tag{45}$$

where $q(t)$ is a continuous and uniformly bounded function. We assume that system (45) has the solution $y(t)$ with the positive lower characteristic exponent ρ:

$$\lim_{t\to+\infty} \inf \frac{1}{t} \ln|y(t)| = \rho > 0. \tag{46}$$

Since system (45) can be represented as

$$\frac{dx}{dt} = -q(t)x, \quad x(t) = y(t)^{-1},$$

from Theorem 2 we obtain that for any $\alpha \in (0, \rho)$ and for a certain $\gamma \geq 0$, estimate (40) is satisfied. Then for $P(t) = -q(t)$ we have the solution $H(t)$ of the Lyapunov equation with properties (42) and (44).

Assuming then $m(t) = H(t)^{-1}$ and $G(t) \equiv 1$, we obtain the following

Corollary 2 *If $\rho > 0$, then there exists a continuously differentiable bounded for $t \geq 0$ positive function $m(t)$ such that*

$$\dot{m}(t) + 2q(t)m(t) = m(t)^2, \quad \forall\, t \geq 0.$$

In the discrete case we similarly have

Corollary 3 *Let the scalar equation*

$$y(t+1) = q(t)y(t) \qquad (45')$$

have the positive lower characteristic exponent ρ :

$$\rho = \lim_{t\to+\infty} \inf \frac{1}{t} \ln \Big| \prod_{j=0}^{t-1} q(j) \Big| > 0. \qquad (46')$$

Then there exists a positive bounded for $t \geq 0$ *function* $m(t)$ *such that*

$$q(t)m(t+1)q(t) - m(t) = q(t)^2 m(t+1)m(t) > 0, \quad \forall\, t = 0, 1, 2\ldots .$$

Proof. Represent Eq. (45′) as

$$x(t+1) = p(t)x(t), \quad x(t) = y(t)^{-1},\ p(t) = q(t)^{-1}.$$

Then from (46′) we obtain

$$\mathcal{X}[x(t)] = \lim_{t\to+\infty} \sup \frac{1}{t} \ln \Big| \prod_{j=0}^{t-1} p(t) \Big| = - \lim_{t\to+\infty} \inf \frac{1}{t} \ln \Big| \prod_{j=0}^{t-1} q(j) \Big| = -\rho < 0.$$

Note that for any $\alpha \in (0, \rho)$ and for a certain $\gamma \geq 0$, estimates (40) are satisfied, namely

$$|x(t)x(\tau)^{-1}| \ \leq \ Ce^{-\alpha(t-\tau)+\gamma\tau}, \quad t \geq \tau,$$

where $\alpha = \rho - \varepsilon$, $\gamma = \Gamma + \varepsilon$ (see Theorem 2). Then for $p(t) = q(t)^{-1}$ there exists the solution of the Lyapunov equation $h(t)$ with properties (42), (44):

$$|h(t)| \leq Re^{2\gamma t}, \ \ z^* h(t) z \geq \varepsilon |z|^2.$$

Put $m(t) = h(t)^{-1}$ and $G(t) \equiv 1$. Then from (39′) we obtain assertion of the Corollary. ■

By corollaries of Theorem 3, for the first-order equation we can easily prove [Leonov, 2008] the following

Preposition 2 *For the first-order equation with bounded coefficients the positiveness of lower characteristic exponent of the first approximation system results in the exponential instability of zero solution of the original system.*

1.6 Stability criteria by the first approximation

1.6.1 The lemmas of Bellman — Gronwall, Bihari and their discrete analogs

For the proof of theorems on stability by the first approximation, the lemmas of Bellman — Gronwall, Bihari [Demidovich, 1967] and their discrete analogs [Demidovich, 1969, Kuznetsov & Leonov, 2005[2]] are often used.

Lemma 6 *(Bellman — Gronwall) If for the non-negative continuous functions $u(t)$ and $v(t)$ for $t \geq 0$ and for a number $C > 0$ the following inequality*

$$u(t) \leq C + \int_0^t v(\tau)u(\tau)\,d\tau, \quad \forall t \geq 0 \tag{47}$$

holds, then the estimate

$$u(t) \leq C \exp\left(\int_0^t v(\tau)\,d\tau\right), \quad \forall t \geq 0 \tag{48}$$

is valid.

Lemma 7 *(Bihari) If for the non-negative continuous functions $u(t)$ and $v(t)$ for $t \geq 0$ and for certain numbers $\nu > 1$ and $C > 0$ the following inequalities*

$$u(t) \leq C + \int_0^t v(\tau)\big(u(\tau)\big)^{\nu}\,d\tau, \quad \forall t \geq 0, \tag{49}$$

$$(\nu - 1)C^{\nu-1} \int_0^t v(\tau)\,d\tau < 1, \quad \forall t \geq 0 \tag{50}$$

hold, then the estimate

$$u(t) \leq C \left(1 - (\nu - 1)C^{\nu-1} \int_0^t v(\tau)\,d\tau\right)^{-1/(\nu-1)}, \quad \forall t \geq 0 \tag{51}$$

is valid.

Proof. We introduce the function $\Phi(u) = u$ for the proof of Lemma 6 and the function $\Phi(u) = u^{\nu}$ for the proof of Lemma 7. Since Φ increases, from inequalities (47) and (49) we have the following estimate

$$\Phi\big(u(t)\big) \leq \Phi\big(C + \int_0^t v(\tau)\Phi(u(\tau))\,d\tau\big).$$

This implies the inequality

$$\frac{v(t)\Phi(u(t))}{\Phi\left(C + \int_0^t v(\tau)\Phi(u(\tau))\,d\tau\right)} \leq v(t).$$

Setting $w(t) = C + \int\limits_0^t v(\tau)\Phi(u(\tau))\,d\tau$ and integrating the last inequality from 0 to t, we obtain the estimate

$$\int\limits_0^t \frac{\dot{w}(s)}{\Phi(w(s))}\,ds \leq \int\limits_0^t v(s)\,ds.$$

For $\Phi(u) = u$ this inequality can be represented as

$$\frac{w(t)}{w(0)} \leq \exp\left(\int\limits_0^t v(s)\,ds\right)$$

and for $\Phi(u) = u^\nu$ as

$$\frac{1}{1-\nu}\left(\frac{1}{w(t)^{\nu-1}} - \frac{1}{w(0)^{\nu-1}}\right) \leq \int\limits_0^t v(s)\,ds.$$

Taking into account the conditions $u(t) \leq w(t)$, $w(0) = C$ and (50), we obtain the assertions of Lemmas 6 and 7. ■

Consider now the discrete analogs of these lemmas [Kuznetsov & Leonov, 2005[2]].

Lemma 8 *If for the non-negative sequences* $\{u(t)\}_{t=0}^{\infty}$ *and* $\{v(t)\}_{t=0}^{\infty}$ *there exists a number* $C > 1$ *such that the following condition*

$$u(t) \leq Cu(0) + \sum_{n=0}^{t-1} v(n)u(n), \quad t = 1, 2... \tag{52}$$

is satisfied, then the inequality

$$u(t) \leq Cu(0)\prod_{n=0}^{t-1}(v(n)+1), \quad t = 1, 2... \tag{53}$$

is valid.

Proof. Consider the sequence

$$\widetilde{u}(t) = \widetilde{u}(0) + \sum_{n=0}^{t-1} v(n)\widetilde{u}(n), \quad \widetilde{u}(0) = Cu(0).$$

By the assumption $C > 1$ and condition (52) we have the estimate $u(t) \leq \widetilde{u}(t)$. Then from the form of $\widetilde{u}(t)$ it follows that

$$\widetilde{u}(t+1) = \widetilde{u}(0) + \sum_{n=0}^{t-1} v(n)\widetilde{u}(n) + v(t)\widetilde{u}(t) = \widetilde{u}(t)(v(t)+1).$$

Hence

$$u(t) \leq \widetilde{u}(t) = \widetilde{u}(0)\prod_{n=0}^{t-1}(v(n)+1), \quad t = 1, 2\ldots.$$

Lemma is proved. ■

Corollary 4 *If the sequences* $\{u(t)\}_{t=0}^{\infty}$, $\{v(t)\}_{t=0}^{\infty}$ *satisfy the conditions of Lemma 8, then*

$$u(t) \leq Cu(0)\ \exp\big(\sum_{n=0}^{t-1} v(n)\big). \tag{54}$$

Proof. The proof follows directly from the inequality $1+t \leq e^t$ and (53). ■

Corollary 5 *Suppose, for the non-negative sequence* $\{u(t)\}_{t=0}^{\infty}$ *there exist numbers* $C \geq 1,\ 0 \leq r < 1,\ m > 1$ *and* $C_r \geq 0$ *such that the following condition*

$$u(t) \leq Cu(0) + \sum_{n=0}^{t-1} C_r r^n u(n)^m, \quad t = 1, 2... \tag{55}$$

is satisfied. Then for sufficiently small $u(0)$ *the inequality*

$$u(t) < 1, \quad t = 0, 1...$$

is satisfied.

Proof.
By the assumption for the sequence $v(t) = C_r r^t$ we have $\sum_{n=0}^{+\infty} v(n) < +\infty$. Put

$$u(t)' = Cu(0)' + \sum_{n=0}^{t-1} v(t)u(n)'.$$

By Corollary 4 we can choose $u(0)'$ such that $u(t)' < 1$ for all t.

A further proof is by induction. Assuming $u(0) < u(0)'$, by (55) we have

$$u(1) \leq Cu(0) + v(0)u(0)^m \leq Cu(0)' + v(0)u(0)' = u(1)' < 1.$$

If $u(t)' < 1$, then

$$u(t+1) \leq Cu(t) + v(t)u(t)^m \leq Cu(t)' + v(t)u(t)' = u(t+1)' < 1.$$

■

1.6.2 Stability criteria by the first approximation

Represent the solutions of systems (3) and (3′) in the Cauchy form. In the continuous case we have

$$x(t) = X(t)x(0) + \int_0^t X(t)X(\tau)^{-1} f\big(\tau, x(\tau)\big)\, d\tau, \tag{56}$$

and in the discrete one

$$x(t) = X(t)x(0) + \sum_{\tau=0}^{t-1} X(t)X(\tau+1)^{-1} f\big(\tau, x(\tau)\big),\ t = 1, 2.... \tag{57}$$

Here $X(t)$ is a fundamental matrix of the linear part of the system.

Recall that by condition (4) the nonlinear part $f(t,x)$ of systems (3) and (3′) in a certain neighborhood $\Omega(0)$ of the point $x=0$ satisfies the following condition

$$|f(t,x)| \le \kappa |x|^{\nu} \ \forall t \ge 0, \quad \forall x \in \Omega(0), \quad \kappa > 0, \nu > 1.$$

We now describe the most famous stability criteria for the solution $x(t) \equiv 0$ by the first approximation.

Consider the continuous case. We assume that there exists a number $C > 0$ and a piecewise continuous function $p(t)$ such that for the Cauchy matrix $X(t)X(\tau)^{-1}$ the estimate

$$|X(t)X(\tau)^{-1}| \le C \exp \int\limits_{\tau}^{t} p(s)\,ds, \quad \forall t \ge \tau \ge 0 \tag{58}$$

is valid.

Theorem 4 *[Leonov, 2008] If condition* (4) *with* $\nu = 1$ *and the inequality*

$$\lim_{t\to+\infty} \sup \frac{1}{t} \int\limits_{0}^{t} p(s)\,ds + C\kappa < 0$$

are satisfied, then the solution $x(t) \equiv 0$ *of system* (3) *is asymptotically Lyapunov stable.*

Proof. From (56) and the hypotheses of theorem we have

$$|x(t)| \le C \exp\left(\int\limits_{0}^{t} p(s)\,ds\right) |x(0)| + C \int\limits_{0}^{t} \exp\left(\int\limits_{\tau}^{t} p(s)\,ds\right) \kappa |x(\tau)|\,d\tau.$$

This estimate can be rewritten as

$$\exp\left(-\int\limits_{0}^{t} p(s)\,ds\right) |x(t)| \le C|x(0)| + C\kappa \int\limits_{0}^{t} \exp\left(-\int\limits_{0}^{\tau} p(s)\,ds\right) |x(\tau)|)\,d\tau.$$

By Lemma 6 the following estimate

$$|x(t)| \le C|x(0)| \exp\left(\int\limits_{0}^{t} p(s)\,ds + C\kappa t\right), \quad \forall t \ge 0$$

is satisfied. This completes the proof of theorem. ∎

We now consider a discrete analog of this theorem. In the discrete case we assume that in place of inequality (58) we have

$$|X(t)X(\tau)^{-1}| \le C \prod_{s=\tau}^{t-1} p(s), \quad \forall t > \tau \ge 0, \tag{59}$$

where $p(s)$ is a positive function.

Theorem 5 *If condition* (4) *with* $\nu = 1$ *and the inequality*

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln \prod_{s=0}^{t-1} \big(p(s) + C\kappa\big) < 0 \tag{60}$$

are satisfied, then the solution $x(t) \equiv 0$ *of system* (3′) *is asymptotically Lyapunov stable.*

Proof. By (57) and (59) we have

$$|x(t)| \le C \prod_{s=0}^{t-1} p(s)|x(0)| + C \sum_{\tau=0}^{t-1} \left(\prod_{s=\tau+1}^{t-1} p(s) \right) \kappa |x(\tau)|.$$

This estimate can be rewritten as

$$|x(t)| \prod_{s=0}^{t-1} p(s)^{-1} \le C|x(0)| + C\kappa \sum_{\tau=0}^{t-1} p(\tau)^{-1} |x(\tau)| \prod_{s=0}^{\tau-1} p(s)^{-1}. \tag{61}$$

Applying Lemma 8 to the relations

$$u(t) = |x(t)| \prod_{s=0}^{t-1} p(s)^{-1},\ u(0) = x(0), \quad v(t) = C\kappa p(t)^{-1},$$

we obtain the following estimate

$$|x(t)| \prod_{s=0}^{t-1} p(s)^{-1} < C|x(0)| \prod_{s=0}^{t-1} \big(C\kappa p(s)^{-1} + 1\big)$$

or, the same,

$$|x(t)| < C|x(0)| \prod_{\tau=0}^{t-1} \big(C\kappa + p(\tau)\big).$$

These estimates and condition (60) prove the theorem. ■

Corollary 6 *For the first-order system the negativeness of characteristic exponent of the first approximation system implies the asymptotic stability of zero solution.*

We now assume that for the Cauchy matrix $X(t)X(\tau)^{-1}$ the following estimate

$$|X(t)X(\tau)^{-1}| \le C \exp\big(-\alpha(t-\tau) + \gamma\tau\big), \quad \forall t \ge \tau \ge 0, \tag{62}$$

where $\alpha > 0$, $\gamma \ge 0$, is satisfied.

Theorem 6 *[Chetaev, 1955, Malkin, 1966, Massera, 1956] Let condition* (4) *with sufficiently small* κ *and condition* (62) *be valid. Then if the inequality*

$$(\nu - 1)\alpha - \gamma > 0 \tag{63}$$

holds, then the solution $x(t) \equiv 0$ *is asymptotically Lyapunov stable.*

Theorem 6 strengthens the well-known Lyapunov theorem on stability by the first approximation for regular systems [Lyapunov, 1892].

Proof. Consider the continuous case. Relations (4), (62) and (56) yield the estimate

$$|x(t)| \le Ce^{-\alpha t}|x(0)| + C\int_0^t \exp\big(-\alpha(t-\tau)+\gamma\tau\big)\kappa|x(\tau)|^{\nu}\,d\tau.$$

This inequality can be rewritten as

$$(e^{\alpha t}|x(t)|) \le C|x(0)| + C\kappa\int_0^t \exp\big(((1-\nu)\alpha+\gamma)\tau\big)\big(e^{\alpha\tau}|x(\tau)|\big)^{\nu}\,d\tau.$$

By Lemma 7 we have the following estimate

$$|x(t)| \le C|x(0)|\exp\big(-\alpha t\big)\big[1-(\nu-1)\big(C|x(0)|\big)^{(\nu-1)}(C\kappa)\times$$

$$\times\int_0^t \exp\big(((1-\nu)\alpha+\gamma)\tau\big)\,d\tau\big]^{-1/(\nu-1)}, \quad \forall\, t\ge 0.$$

From this estimate and condition (63) for sufficiently small $|x(0)|$ we obtain the assertion of the theorem. ■

Now we prove this theorem in the discrete case.

Proof. From (62) we have

$$|X(t)X(\tau+1)^{-1}| \le C\exp\big(-\alpha(t-\tau)+\gamma\tau\big)e^{\alpha+\gamma}$$

By estimate (4) from (57) we obtain

$$|x(t)| \le Ce^{-\alpha t}|x(0)| + C\sum_{\tau=0}^{t-1}\exp\big(-\alpha(t-\tau)+\gamma\tau\big)e^{\alpha+\gamma}\kappa|x(\tau)|^{\nu}.$$

This estimate can be rewritten in the form

$$(e^{\alpha t}|x(t)|) \le C|x(0)| + Ce^{\alpha+\gamma}\kappa\sum_{\tau=0}^{t-1}e^{(\alpha(1-\nu)+\gamma)\tau}(e^{\alpha\tau}|x(\tau)|)^{\nu}.$$

Since by condition (63) of theorem the inequality $(\alpha(1-\nu)+\gamma) < 0$ holds, by Corollary 5 of Lemma 8 for sufficiently small $|x(0)|$ we obtain

$$(e^{\alpha t}|x(t)|) \le 1.$$

Theorem is proved. ■

1.6.3 Stability criteria for the flow and cascade of solutions

Consider a continuous system

$$\frac{dx}{dt} = F(x,t), \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{64}$$

and the discrete one

$$x(t+1) = F\big(t, x(t)\big), \quad x(t) \in \mathbb{R}^n, \quad t = 0,1,2,..., \tag{64$'$}$$

where $F(\cdot,\cdot)$ is a twice continuously differentiable vector-function.

Consider the linearizations of these systems along solutions with the initial data $y = x(0,y)$ from the open set Ω, which is bounded in $\mathbb{R}^n$

$$\frac{dz}{dt} = A_y(t)z, \tag{65}$$

$$z(t+1) = A_y(t)z(t). \tag{65$'$}$$

Here the matrix

$$A_y(t) = \frac{\partial F(x,t)}{\partial x}|_{x=x(t,y)}$$

is Jacobian matrix of the vector-function $F(x,t)$ on the solution $x(t,y)$. Let $X(t,y)$ be a fundamental matrix of linear system and $X(0,y) = I_n$.

We assume that for the largest singular value $\alpha_1(t,y)$ of systems (65) and (65′) for all t the following estimate

$$\alpha_1(t,y) < \alpha(t), \quad \forall y \in \Omega, \tag{66}$$

where $\alpha(t)$ is a scalar function, is valid.

Theorem 7 *[Leonov, 1998, Kuznetsov & Leonov, 2005[2]] Suppose, the function $\alpha(t)$ is bounded on the interval $(0,+\infty)$. Then the flow (cascade) of solutions $x(t,y)$, $y \in \Omega$, of systems* (64) *and* (64′) *are Lyapunov stable.*

If, in addition,

$$\lim_{t\to+\infty} \alpha(t) = 0,$$

then the flow (cascade) of solutions $x(t,y)$, $y \in \Omega$, is asymptotically Lyapunov stable.

Proof. It is well known that

$$\frac{\partial x(t,y)}{\partial y} = X(t,y), \quad \forall t \geq 0.$$

It is also known [Zorich, 1984] that for any vectors y, z, and a number $t \geq 0$ there exists a vector w such that the relations

$$|w - y| \leq |y - z|,$$

$$|x(t,y) - x(t,z)| \leq \left|\frac{\partial x(t,w)}{\partial w}\right| |y - z|$$

are satisfied. Therefore for any vector z from the ball centered at y and placed entirely in Ω the following estimate

$$|x(t,y) - x(t,z)| \leq |y - z| \sup \alpha_1(t,w) \leq \alpha(t)|y - z|, \quad \forall t \geq 0 \tag{67}$$

is valid. Here the supremum is taken over all w from the ball $\{w : |w - y| \leq |y - z|\}$.
Estimate (67) gives at once the assertions of theorem. ■

Corollary 7 *The Perron effects are possible on the boundary of the stable by the first approximation solutions flow (cascade) only.*

Consider the flow of solutions of system (25) with the initial data in a neighborhood of the point $x_1 = x_2 = 0$: $x_1(0, x_{10}, x_{20}) = x_{10}$, $x_2(0, x_{10}, x_{20}) = x_{20}$.
Hence it follows easily that

$$x_1(t, x_{10}, x_{20}) = \exp\big(-at\big)x_{10}.$$

Therefore for continuous system the matrix $A(t)$ of linear system takes the form

$$A(t) = \begin{pmatrix} -a & 0 \\ 2\exp\big(-at\big)x_{10} & r(t) \end{pmatrix}, \tag{68}$$

where

$$r(t) = \sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a.$$

For the discrete system we have

$$A(t) = \begin{pmatrix} e^{-a} & 0 \\ 2\exp\big(-at\big)x_{10} & r(t) \end{pmatrix}, \tag{68$'$}$$

$$r(t) = \frac{\exp\big((t+2)\sin\ln(t+2) - 2a(t+1)\big)}{\exp\big((t+1)\sin\ln(t+1) - 2at\big)}.$$

The solutions of system (65) and (65′) with matrices (68) and (68′), respectively, are the following

$$\begin{aligned} z_1(t) &= \exp\big(-at\big)z_1(0), \\ z_2(t) &= p(t)(z_2(0) + 2x_{10}z_1(0))q(t)). \end{aligned} \tag{69}$$

Here

$$p(t) = \exp\big((t+1)\sin(\ln(t+1)) - 2at\big),$$

$$q(t) = \int_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)\,d\tau$$

in the continuous case and

$$p(t) = \exp\big((t+1)\sin(\ln(t+1)) - 2at\big),$$

$$q(t)=\sum_{k=0}^{t-1}\exp\big(-(k+2)\sin\ln(k+2)+2a\big)$$

in the discrete case.

As was shown above (29), if relations (26) are satisfied and

$$z_1(0)x_{10}\neq 0,$$

then the characteristic exponent of $z_2(t)$ is positive.

Hence in an arbitrary small neighborhood of the trivial solution $x_1(t)\equiv x_2(t)\equiv 0$ there exist the initial data x_{10},x_{20} such that for $x_1(t,x_{10},x_{20}),x_2(t,x_{10},x_{20})$ the first approximation system has the positive largest characteristic exponent (and the Lyapunov exponent μ_1).

Therefore in this case there does not exist a neighborhood Ω of the point $x_1=x_2=0$ such that uniform estimates (66) are satisfied. Thus, for systems (25), (25′) the Perron effect occurs.

1.7 Instability Criteria by the First Approximation

1.7.1 The Perron — Vinograd triangulation method

One of basic procedures for analysis of instability is a reduction of the linear part of the system to the triangular form. In this case the Perron — Vinograd triangulation method for a linear system [Demidovich, 1967, Bylov *et al.*, 1966, Coppel, 1978] turns out to be most effective. It will be described below.

Let

$$Z(t)=\big(z_1(t),...,z_n(t)\big)$$

be a fundamental system of solutions of linear continuous system (5) or discrete system (5′).

We apply the Schmidt orthogonalization procedure to the solutions $z_j(t)$.

$$\begin{aligned}
&v_1(t)=z_1(t)\\
&v_2(t)=z_2(t)-v_1(t)^*z_2(t)\frac{v_1(t)}{|v_1(t)|^2}\\
&\dots\dots\dots\dots\dots\\
&v_n(t)=z_n(t)-v_1(t)^*z_n(t)\frac{v_1(t)}{|v_1(t)|^2}-\ldots-v_{n-1}(t)^*z_n(t)\frac{v_{n-1}(t)}{|v_{n-1}(t)|^2}.
\end{aligned}\tag{70}$$

Relations (70) yield the following relations

$$v_i(t)^*v_j(t)=0,\quad \forall j\neq i,\tag{71}$$

$$|v_j(t)|^2=v_j(t)^*z_j(t).\tag{72}$$

If for the fundamental matrix $Z(t)$ the relation $Z(0) = I_n$ holds, we conclude that $V(0) = \big(v_1(0), .., v_n(0)\big) = I_n$.

From (72) we have the following

Lemma 9 *The following estimate*

$$|v_j(t)| \leq |z_j(t)|, \quad \forall t \geq 0 \tag{73}$$

is valid.

We proceed now to the description of *the triangulation procedure of Perron — Vinograd.*

Consider the unitary matrix

$$U(t) = \left(\frac{v_1(t)}{|v_1(t)|}, \cdots, \frac{v_n(t)}{|v_n(t)|} \right),$$

and make the change of variable: $z = U(t)w$ in the linear system. In the continuous case we obtain the system

$$\frac{dw}{dt} = B(t)w, \tag{74}$$

where

$$B(t) = U(t)^{-1}A(t)U(t) - U(t)^{-1}\dot{U}(t), \tag{75}$$

and in the discrete case the system

$$w(t+1) = B(t)w(t), \tag{74'}$$

where

$$B(t) = U(t+1)^{-1}A(t)U(t). \tag{75'}$$

The unitarity of the matrix $U(t)$ implies that for the columns $w(t)$ of the fundamental matrix

$$W(t) = \big(w_1(t), \ldots, w_n(t)\big) = U(t)^* Z(t), \tag{76}$$

the relations $|w_j(t)| = |z_j(t)|$ are satisfied.

By (70)–(72) we obtain that the matrix $W(t)$ has the upper triangular form with the diagonal elements $|v_1(t)|$,..,$|v_n(t)|$, namely

$$W(t) = \begin{pmatrix} |v_1(t)| & \cdots & \\ & \ddots & \vdots \\ 0 & & |v_n(t)| \end{pmatrix}. \tag{77}$$

From the fact that $W(t)$ is an upper triangular matrix it follows that $W(t)^{-1}$, $\dot{W}(t)$ are also upper triangular matrices. Hence $B(t)$ is an upper triangular matrix with the diagonal elements $b_1(t)$,..,$b_n(t)$:

$$B(t) = \begin{pmatrix} b_1(t) & \cdots & \\ & \ddots & \vdots \\ 0 & & b_n(t) \end{pmatrix}, \tag{78}$$

where in the continuous case $b_i(t) = (\ln|v_i(t)|)^\bullet$ and in the discrete one

$$b_i(t) = \frac{|v_i(t+1)|}{|v_i(t)|}.$$

Thus, we proved the following

Theorem 8 *(Perron triangulation [Demidovich, 1967, Kuznetsov & Leonov, 2005[1]]) By means of the unitary transformation $z = U(t)w$ the linear system can be reduced to the linear system with the upper triangular matrix $B(t)$.*

Note that if $|A(t)|$ is bounded for $t \geq 0$, then $|B(t)|$, $|U(t)|$, and $|\dot{U}(t)|$ are also bounded for $t \geq 0$. If in the discrete case, in addition, $|A(t)^{-1}|$ is bounded for $t \geq 0$, then $|B(t)^{-1}|$ is also bounded for $t \geq 0$.

Define the vector $z_i' = z_i - v_i$. Then the vector z_i' is orthogonal to the vector v_i, where $i \geq 2$. Consider the angle included between the vectors z_i and z_i'. Note that from definition of the angle included between the vectors we have $\angle(z_i, z_i') \leq \pi$. In this case the following relation

$$|v_i| = |z_i| \sin\big(\angle(z_i, z_i')\big) \quad i \geq 2 \tag{79}$$

is valid.

By (79) from (76) and (77) we have

$$|\det Z(t)| = |\det U(t)| \prod_{i=1}^{n} |v_i| = \prod_{i=1}^{n} |z_i| \prod_{k=2}^{n} |\sin\big(\angle(z_k, z_k')\big)|.$$

With the help of this relation in [Vinograd, 1954] the following criterion of system regularity was obtained.

Theorem 9 *[Vinograd, 1954]*

Consider a linear system with bounded coefficients and its certain fundamental system of solutions $Z(t) = \big(z_1(t), ..., z_n(t)\big)$. Let there exist the exact characteristic exponents of $|z_i(t)|$

$$\lim_{t\to+\infty} \frac{1}{t} \ln|z_i(t)| \quad i = 1, ..., n \tag{80}$$

and let there exist and be equal to zero the exact characteristic exponents of sines of the angles $\angle(z_i, z_i')$

$$\lim_{t\to+\infty} \frac{1}{t} \ln\big|\sin\big(\angle(z_i, z_i')\big)\big| = 0 \quad i = 2, ..., n\,. \tag{81}$$

Then the linear system is regular and $Z(t)$ is a normal system of solutions.

Conversely, if the linear system is regular and $Z(t)$ is a normal system of solutions, then (80) and (81) are satisfied.

We now obtain another useful estimate for the vector-function $v_j(t)$. We might ask how far the vector-function $v_j(t)$ can decrease in comparison with the original system of the vectors $z_j(t)$. The answer is due to the following

Lemma 10 *If for a certain number C the inequality*

$$\prod_{j=1}^{n} |z_j(t)| \leq C|\det Z(t)| \quad \forall\, t \geq 0 \tag{82}$$

is valid, then there exists a number $r > 0$ such that the following estimate

$$|z_j(t)| \leq r|v_j(t)|, \quad \forall t \geq 0, \;\; j = 1, \ldots, n \tag{83}$$

is satisfied.

Proof. The Ostrogradsky — Liouville formula [Demidovich, 1967] and inequality (82) give

$$\left|\det\left(\tfrac{z_1(t)}{|z_1(t)|}, \ldots, \tfrac{z_n(t)}{|z_n(t)|}\right)\right| =$$
$$= \big(|z_1(t)|, \ldots, |z_n(t)|\big)^{-1}\big|\det\big(z_1(0), \ldots, z_n(0)\big)\big|\big|\det Z(t)\big| \geq$$
$$\geq C^{-1}\big|\det\big(z_1(0), \ldots, z_n(0)\big)\big|, \;\; \forall\, t \geq 0.$$

This implies that for ($1 \leq m < n$) for the linear subspace $L(t)$, spanned on the vectors $z_1(t), \ldots, z_m(t)$, a number $\varepsilon \in (0, 1)$ can be found such that the estimate

$$\frac{|z_{m+1}(t)^* e(t)|}{|z_{m+1}(t)|} \leq 1 - \varepsilon, \quad \forall t \geq 0 \tag{84}$$

is valid for all $e(t) \in L(t)$, satisfying the relation $|e(t)| = 1$.
Relations (70) can be rewritten as

$$\frac{v_j(t)}{|z_j(t)|} = \prod_{i=1}^{j-1}\left(I_n - \frac{v_i(t)v_i(t)^*}{|v_i(t)|^2}\right)\frac{z_j(t)}{|z_j(t)|}. \tag{85}$$

Suppose, the lemma is false. Then there exists a sequence $t_k \to +\infty$ such that

$$\lim_{k\to+\infty} \frac{v_j(t_k)}{|z_j(t_k)|} = 0.$$

However by (85) we obtain that there exists a number $l < j$ such that

$$\lim_{k\to+\infty}\left[\frac{z_j(t_k)}{|z_j(t_k)|} - \frac{v_l(t_k)}{|v_l(t_k)|}\right] = 0. \tag{86}$$

Since $v_l(t) \in L(t)$, relations (84) and (86) are incompatible. This contradiction proves the estimate (83). ■

Corollary 8 *Condition* (82) *is necessary and sufficient for the existence of the number $r > 0$ such that estimate* (83) *holds.*

Note that condition (82) is necessary and sufficient for the nondegenaracy of normalized fundamental matrix of the first approximation system, as $t \to +\infty$,:

$$\lim_{t\to+\infty} \inf\left|\det\left(\frac{z_1(t)}{|z_1(t)|}, \cdots, \frac{z_n(t)}{|z_n(t)|}\right)\right| > 0.$$

Lemma 11 *The following estimate*

$$\frac{|v_n(t)|}{|v_n(\tau)|} \geq \frac{|\det Z(t)|}{|\det Z(\tau)|} \prod_{j=1}^{n-1} \frac{|v_j(\tau)|}{|z_j(t)|}, \quad \forall t, \tau \geq 0 \tag{87}$$

is valid.

Proof. Relation (77) yields the relation

$$\frac{|v_n(t)|}{|v_n(\tau)|} = \frac{|\det W(t)| \prod_{j=1}^{n-1} |v_j(\tau)|}{|\det W(\tau)| \prod_{j=1}^{n-1} |v_j(t)|}.$$

From the unitarity of the matrix $U(t)$ and from (76) we have

$$|\det Z(t)| = |\det W(t)|.$$

■

1.7.2 Instability criterion by Krasovsky

Consider instability by Krasovsky for the solution $x(t) \equiv 0$ of continuous system (3) and of discrete system (3′).

In the continuous case the following theorem strengthens the result, obtained in [Leonov, 2002[1], Leonov, 2004]. In the discrete case this result is reduced to that, obtained in [Kuznetsov & Leonov, 2005[1]].

Theorem 10 *If the relation*

$$\sup_{1 \leq k \leq n} \lim_{t \to +\infty} \inf \left[\frac{1}{\ln t} \left(\ln |\det Z(t)| - \sum_{j \neq k} \ln |z_j(t)| \right) \right] > 1 \tag{88}$$

is satisfied, then the solution $x(t) \equiv 0$ is unstable by Krasovsky.

Proof. We can assume, without loss of generality, that in (88) the supremum, taken over k, is attained for $k = n$. Then by Lemma 11 from condition (88) we obtain that there exists a number $\mu > 1$ such that for sufficiently large t the following estimate

$$\ln |v_n(t)| \geq \mu \ln t, \quad \mu > 1 \tag{89}$$

holds. Suppose now that the solution $x(t) \equiv 0$ is stable by Krasovsky. This means that in a certain neighborhood of the point $x = 0$ there exists a number $R > 0$ such that the estimate

$$|x(t, x_0)| \leq R|x_0|, \quad \forall t \geq 0 \tag{90}$$

is valid. We make use of the Perron — Vinograd change of variable

$$x = U(t)y \tag{91}$$

to obtain a system with the upper triangular matrix $B(t)$ of the type (78).
1. Consider the continuous case. Using (91), from continuous system (3) we obtain

$$\frac{dy}{dt} = B(t)y + g(t,y), \tag{92}$$

where

$$g(t,y) = U(t)^{-1} f\big(t, U(t)y\big).$$

Thus, the last equation of system (92) takes the form

$$\frac{dy_n}{dt} = \big(\ln|v_n(t)|\big)^{\bullet} y_n + g_n(t,y). \tag{93}$$

Here y_n and g_n are the nth components of the vectors y and g, respectively. Conditions (4) and (90) yield the estimate

$$\big|g\big(t,y(t)\big)\big| \le \varkappa R^{\nu} \big|y(0)\big|^{\nu}. \tag{94}$$

Note that the solution $y_n(t)$ of Eq. (93) can be represented in the form

$$y_n(t) = \frac{|v_n(t)|}{|v_n(0)|}\left(y_n(0) + \int\limits_0^t \frac{|v_n(0)|}{|v_n(s)|} g\big(s, y(s)\big)\, ds \right). \tag{95}$$

Estimate (89) implies that there exists a number $\rho > 0$ such that the following inequalities

$$\int\limits_0^t \frac{|v_n(0)|}{|v_n(s)|}\, ds \le \rho, \quad \forall t \ge 0 \tag{96}$$

are valid. Now we take the initial condition $x_0 = U(0)y(0)$ in such a way that $y_n(0) = |y(0)| = \delta$, where the number δ satisfies the inequality

$$\delta > \rho \varkappa R^{\nu} \delta^{\nu}. \tag{97}$$

Then from (94)–(96) for sufficiently large $t \ge 0$ we obtain the following estimate

$$y_n(t) \ge t^{\mu}(\delta - \rho\varkappa R^{\nu}\delta^{\nu}), \quad \mu > 1.$$

By (97)

$$\lim_{t\to+\infty} \inf y_n(t) = +\infty.$$

The latter contradicts the assumption on stability by Krasovsky of a trivial solution of system (3).

2. Now we prove the theorem in the discrete case. By (91), from discrete system (3′) we have

$$y(t+1) = B(t)y(t) + g\big(t, y(t)\big), \tag{98}$$

where

$$g\big(t,y(t)\big) = U(t+1)^{-1} f\big(t, U(t)y(t)\big).$$

Then the last equation of system (98) takes the form

$$y_n(t+1) = \frac{|v_n(t+1)|}{|v_n(t)|} y_n(t) + g_n\big(t, y(t)\big), \tag{99}$$

where y_n and g_n are the nth components of the vectors y and g, respectively. Conditions (4) and (90) give the following estimate

$$\big|g\big(t, y(t)\big)\big| \le \varkappa R^\nu \big|y(0)\big|^\nu. \tag{100}$$

Note that the solution $y_n(t)$ of Eq. (99) can be represented as

$$y_n(t) = \frac{|v_n(t)|}{|v_n(0)|} \left(\sum_{j=0}^{t-1} \frac{|v_n(0)|}{|v_n(j+1)|} g_n\big(j, y(j)\big) + y_n(0) \right). \tag{101}$$

Estimate (89) implies that there exists a number $\rho > 0$ such that the following inequality

$$\sum_{j=0}^{t-1} \frac{|v_n(0)|}{|v_n(j+1)|} < \rho, \quad t \ge 1 \tag{102}$$

is satisfied. Taking the same initial data as in the continuous case (97), we obtain

$$\lim_{t\to+\infty} \inf y_n(t) = +\infty.$$

The latter contradicts the assumption on stability by Krasovsky of a trivial solution of system (3′).

This proves the theorem. ■

Remark, concerning the method for the proof of theorem.

Assuming that the zero solution of the considered system is Lyapunov stable and using the same reasoning as in the case of stability by Krasovsky, we need to prove in the continuous case the following inequality

$$y_n(0) + \int_0^{+\infty} \frac{|v_n(0)|}{|v_n(s)|} g\big(s, y(s)\big)\, ds \ne 0. \tag{103}$$

While the above inequality is easily proved in the case of stability by Krasovsky, this becomes an intractable problem in the case of stability by Lyapunov.

A scheme similar to that, considered above for reducing the problem to one scalar equation of the type (93), was used by N.G. Chetaev [1990; 1948] to obtain instability criteria. In the scheme, suggested by N.G. Chetaev for proving inequality (103), a similar difficulty occurs. Therefore, at present, Chetaev's technique permits us to obtain the criteria of instability by Krasovsky only.

The method to obtain the criteria of instability by Lyapunov invites further development. Such development under certain additional restrictions will be presented in Theorem 12.

Condition (88) of Theorem 10 is satisfied if the following inequality

$$\Lambda - \Gamma > 0 \tag{104}$$

is valid.

Here Λ is the largest characteristic exponent, Γ is the irregularity coefficient.

The condition of instability by Krasovsky (104) was obtained under the additional condition of the analyticity of the function $f(t,x)$ by N.G. Chetaev [1990; 1948]

Recall here stability condition (63) of Theorem 6, which by Theorem 2 can be represented as

$$(\nu - 1)\Lambda + \Gamma < 0. \tag{105}$$

Since Theorems 4–6 give, at the same time, the criteria of stability by Krasovsky, we can formulate the following

Theorem 11 *[Leonov, 2004]) If*

$$\Lambda < \frac{-\Gamma}{(\nu - 1)},$$

then the solution $x(t) \equiv 0$ is Krasovsky stable and if

$$\Lambda > \Gamma,$$

then the solution $x(t) \equiv 0$ is Krasovsky unstable.

For regular systems (the case $\Gamma = 0$), Theorem 11 gives a complete solution of the problem of stability by Krasovsky in the noncritical case ($\Lambda \neq 0$).

Note that for system (35) the relation $\Gamma = \Lambda + 2a + 1$ holds. Therefore for system (35) condition (104) is untrue.

1.7.3 Instability criterion by Lyapunov

The Lyapunov instability of solutions of one-dimensional system was considered in Proposition 2. Consider now the Lyapunov instability of the solution $x(t) \equiv 0$ of multidimensional continuous system (3) and of discrete system (3′).

Theorem 12 *[Leonov, 2004, Kuznetsov & Leonov, 2005[1]] Let for certain values $C > 0$, $\beta > 0$, $\alpha_1, ..., \alpha_{n-1}$ ($\alpha_j < \beta$ for $j = 1, \dots, n-1$) the following conditions hold:*

1)

$$|z_j(t)| \leq C \exp(\alpha_j (t - \tau)) |z_j(\tau)|, \qquad \forall\, t \geq \tau \geq 0, \quad j = 1, \dots, n-1, \tag{106}$$

2)

$$\frac{1}{(t-\tau)} \ln |\det Z(t)| > \beta + \sum_{j=1}^{n-1} \alpha_j, \quad \forall\, t \geq \tau \geq 0, \tag{107}$$

and, if $n > 2$,

3)

$$\prod_{j=1}^{n} |z_j(t)| \leq C |\det Z(t)|, \ \ \forall t \geq 0. \tag{108}$$

Then the zero solution of the system considered is Lyapunov unstable.

Proof. Using the Perron — Vinograd triangulation method, we change variables

$$x = U(t)w$$

and separate the last equation (for x_n). Then we obtain a system of $(n-1)$ equations, a fundamental matrix of which is truncated in the upper triangular matrix (77), namely

$$\widetilde{W}(t) = \big(\widetilde{w}_1(t), ..., \widetilde{w}_{n-1}(t)\big) = \begin{pmatrix} w_{11}(t) & \cdots & w_{n-1,1}(t) \\ & \ddots & \\ & & \vdots \\ 0 & & w_{n-1,n-1}(t) \end{pmatrix}.$$

We introduce the following notation for truncated matrix (78)

$$\widetilde{B}(t) = \begin{pmatrix} b_1(t) & & \\ & \ddots & \\ 0 & & b_{n-1}(t) \end{pmatrix}.$$

For $n > 2$ from condition (108) and the identities $|w_j(t)| \equiv |z_j(t)|$, which result from the unitary of the matrix $U(t)$, we obtain the following estimates

$$\begin{array}{c} |\widetilde{w}_j(t)| \le C \exp\big(\alpha_j(t-\tau)\big)|\widetilde{w}_j(\tau)|, \\ \forall t \ge \tau \ge 0, \quad j = 1, \dots, n-1 \end{array}. \tag{109}$$

In addition, by Lemma 10 from condition (108) we obtain estimates (83) and by Lemma 11 from conditions (106) and (107) the estimate

$$\frac{|v_n(t)|}{|v_n(\tau)|} \ge \exp\big(\beta(t-\tau)\big)C^{1-n}\prod_{j=1}^{n-1}\frac{|v_j(\tau)|}{|z_j(\tau)|}, \quad \forall\, t \ge \tau \ge 0. \tag{110}$$

By (83), (110)

$$\frac{|v_n(t)|}{|v_n(\tau)|} \ge \exp\big(\beta(t-\tau)\big)(Cr)^{1-n}, \quad \forall t \ge \tau \ge 0. \tag{111}$$

Since for $n = 2$ the relation $v_1(t) = z_1(t)$ is satisfied, from inequality (110) we obtain a similar estimate

$$\frac{|v_2(t)|}{|v_2(\tau)|} \ge C^{-1}\exp\big(\beta(t-\tau)\big), \quad \forall t \ge \tau \ge 0$$

without assumption (108).

In the original system we make the change of variables

$$x = e^{dt}U(t)y, \tag{112}$$

choosing the number $d > 0$ in such a way that

$$\alpha < d < \beta,$$

where $\alpha = \max \alpha_j,\ j = 1, \dots, n-1$.

Denote by y_k the components of the vector y

$$y = \begin{pmatrix} \widetilde{y} \\ y_n \end{pmatrix}, \quad \widetilde{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

1. In the case of continuous system (3), using (112), we obtain

$$\frac{dy}{dt} = \big(B(t) - dI_n\big)y + g(t, y), \tag{113}$$

where

$$g(t, y) = e^{-dt}U(t)^{-1} f\big(t, e^{dt}U(t)y\big).$$

Condition (4) implies that for any number $\rho > 0$ there exists a neighborhood $\Omega(0)$ of the point $y = 0$ such that

$$|g(t, y)| \leq \rho |y|, \quad \forall t \geq 0,\ \forall y \in \Omega(0). \tag{114}$$

Note that by (109) for the system

$$\frac{d\widetilde{y}}{dt} = \big(\widetilde{B}(t) - dI_{n-1}\big)\widetilde{y}, \quad \widetilde{y} \in \mathbb{R}^{n-1} \tag{115}$$

we have the following estimate

$$|\widetilde{y}(t)| \leq C \exp\big((\alpha - d)(t - \tau)\big)|\widetilde{y}(\tau)|, \ \forall t \geq \tau \geq 0. \tag{116}$$

In this case by Theorem 3 there exist a continuously differentiable matrix $H(t)$ bounded on $[0, +\infty)$ and positive numbers ρ_1 and ρ_2 such that

$$\begin{gathered} \widetilde{y}^* \big(\dot{H}(t) + 2H(\widetilde{B}(t) - dI_{n-1})\big)\widetilde{y} \leq -\rho_1 |\widetilde{y}|^2, \\ \forall \widetilde{y} \in \mathbb{R}^{n-1}, \quad \forall t \geq 0, \end{gathered} \tag{117}$$

$$\widetilde{y}^* H(t)\widetilde{y} \geq \rho_2 |\widetilde{y}|^2, \quad \forall \widetilde{y} \in \mathbb{R}^{n-1},\ \forall t \geq 0. \tag{118}$$

For the scalar equation

$$\frac{dy_n}{dt} = \big((\ln|v_n(t)|)^{\bullet} - d\big)y_n, \quad y_n \in \mathbb{R}^1,$$

from relation (111) for $n \neq 2$ we obtain the following estimate

$$|y_n(t)| \geq (Cr)^{-1} \exp\big((\beta - d)(t - \tau)\big)|y_n(\tau)|, \ \forall t \geq \tau \geq 0.$$

For $n = 2$ a similar estimate takes the form

$$|y_2(t)| \geq C^{-1} \exp\big((\beta - d)(t - \tau)\big)|y_2(\tau)|, \ \forall t \geq \tau \geq 0.$$

Then by the corollary of Theorem 3 there exist a continuously differentiable function $h(t)$ bounded on $[0,+\infty)$ and positive numbers ρ_3 and ρ_4 such that

$$\begin{gathered} \dot{h}(t)+2h(t)\big(\ln|v_n(t)|^\bullet-d\big)\le-\rho_3,\\ h(t)\le-\rho_4,\\ \forall t\ge 0. \end{gathered} \tag{119}$$

We now show that for sufficiently large ω the function

$$V(t,y)=\widetilde{y}^*H(t)\widetilde{y}+\omega h(t)y_n^2$$

is the Lyapunov function, which for system (113) satisfies all conditions of the classical Lyapunov instability theorem [Lyapunov, 1892].

Actually, system (113) can be written in the form

$$\begin{aligned} \frac{d\widetilde{y}}{dt}&=\big(\widetilde{B}(t)-dI_{n-1}\big)\widetilde{y}+q(t)y_n+\widetilde{g}(t,\widetilde{y},y_n),\\ \frac{dy_n}{dt}&=\big((\ln|v_n(t)|)^\bullet-d\big)y_n+g_n(t,\widetilde{y},y_n). \end{aligned} \tag{120}$$

Here $q(t)$ is a certain bounded on $[0,+\infty)$ vector-function, $\widetilde{g}$ and g_n are such that

$$g(t,y)=\begin{pmatrix}\widetilde{g}(t,y)\\ g_n(t,y)\end{pmatrix}.$$

In this case estimates (117), (119) give the inequalities

$$\begin{aligned} V(t,y)^\bullet\le\;& -\rho_1|\widetilde{y}|^2-\omega\rho_3y_n^2+2\widetilde{y}^*H(t)q(t)y_n+2\widetilde{y}^*H(t)\widetilde{g}(t,\widetilde{y},y_n)+\\ &+2\omega h(t)y_ng_n(t,\widetilde{y},y_n)\le\\ \le\;& -\rho_1|\widetilde{y}|^2-\omega\rho_3y_n^2+2\big(|y_n||\widetilde{y}|\sup_t|H(t)|\sup_t|q(t)|+\\ &+|\widetilde{y}|\sup_t|H(t)|\rho(|\widetilde{y}|+|y_n|)+\omega|y_n|\sup_t|h(t)|\rho(|\widetilde{y}|+|y_n|)\big). \end{aligned}$$

From these inequalities and the boundedness of the matrix-function $H(t)$, the vector-function $q(t)$, and the function $h(t)$ it follows that for sufficiently large ω and sufficiently small ρ there exists a number $\theta>0$ such that

$$V(t,y)^\bullet\le-\theta|y|^2. \tag{121}$$

The boundedness of $H(t)$, $h(t)$ implies that there exists a number a such that

$$|y|^2\ge-aV(t,y),\quad\forall t\ge0,\quad\forall y\in\mathbb{R}^n.$$

Then by (121) we have the following inequality

$$V(t,y)^\bullet\le a\theta V(t,y),\quad\forall t\ge0,\quad\forall y\in\mathbb{R}^n. \tag{122}$$

Choose the initial data $y(0)$ in such a way that $V(0,y(0))<0$. Then by (121) we have

$$V\big(t,y(t)\big)<0,\quad\forall\, t\ge0$$

and by inequality (122) the following estimate

$$-V\big(t, y(t)\big) \geq e^{a\theta t}\big(- V\big(0, y(0)\big)\big).$$

In this case inequalities (118) and (119) give the estimate

$$-\omega h(t) y_n(t)^2 \geq e^{a\theta t}\big(- V\big(0, y(0)\big)\big), \quad \forall t \geq 0.$$

Then

$$y_n(t)^2 \geq \frac{e^{a\theta t}}{\omega \sup\limits_{t} (-h(t))}\big(- V\big(0, y(0)\big)\big). \tag{123}$$

This implies the Lyapunov instability of the solution $y(t) \equiv 0$. In addition, from estimate (123) it follows that in a neighborhood of the point $y = 0$ the solution $y(t)$ with the initial data $V\big(0, y(0)\big) < 0$ increases exponentially.

Since $d > 0$ and $U(t)$ is a unitary matrix, the zero solution of system (3) is also Lyapunov unstable.

2. By (112) for discrete system (3′) we obtain the following system

$$y(t+1) = \Big(B(t)e^{-d}\Big)\, y(t) + g\big(t, y(t)\big), \tag{124}$$

where

$$g\big(t, y(t)\big) = e^{-d(t+1)} U^{-1}(t+1) f\big(t, e^{dt} U(t) y(t)\big).$$

Condition (4) implies that for any number $\rho > 0$ there exists a neighborhood $\Omega(0)$ of the point $y = 0$ such that

$$|g(t, y)| \leq \rho |y|, \quad \forall t \geq 0, \ \forall y \in \Omega(0). \tag{125}$$

Note that by (109) for the following system

$$\widetilde{y}(t+1) = \Big(\widetilde{B}(t)e^{-d}\Big)\, \widetilde{y}(t) \tag{126}$$

we have the estimate

$$|\widetilde{y}(t)| \leq C \exp\big((\alpha - d)(t - \tau)\big) |\widetilde{y}(\tau)|, \ \forall\, t \geq \tau \geq 0. \tag{127}$$

Then by Theorem 3 there exist a bounded on $[0, +\infty)$ matrix $H(t)$ and positive values ρ_1 and ρ_2 such that the following relations

$$\widetilde{y}^* \Big(\widetilde{B}(t)^* e^{-d} H(t+1) \widetilde{B}(t) e^{-d} - H(t)\Big)\, \widetilde{y} \leq -\rho_1 |\widetilde{y}|^2, \tag{128}$$

$$\widetilde{y}^* H(t) \widetilde{y} \geq \rho_2 |\widetilde{y}|^2, \quad \forall t \geq 0, \quad \forall \widetilde{y} \in \mathbb{R}^{n-1} \tag{129}$$

are satisfied. From relation (111) for the scalar equation

$$y_n(t+1) = \frac{|v_n(t+1)|}{|v_n(t)|} e^{-d} y_n(t),$$

for $n \neq 2$ we obtain the estimate

$$|y_n(t)| \geq (Cr)^{-1} e^{(\beta-d)(t-\tau)} |y_n(\tau)|, \quad \forall\, t \geq \tau \geq 0.$$

For $n = 2$ a similar estimate is as follows

$$|y_2(t)| \geq (C)^{-1} e^{(\beta-d)(t-\tau)} |y_2(\tau)|, \quad \forall\, t \geq \tau \geq 0.$$

In this case by corollary 2 there exists a continuously differentiable function $h(t)$ bounded on $[0, +\infty)$ and positive numbers ρ_3 and ρ_4 such that

$$y_n(t)^2 \left(\frac{|v_n(t+1)|}{|v_n(t)|} e^{-d} h(t+1) \frac{|v_n(t+1)|}{|v_n(t)|} e^{-d} - h(t) \right) \leq -\rho_3 y_n(t)^2, \tag{130}$$

$$h(t) \leq -\rho_4, \quad \forall\, t \geq 0, \quad \forall\, y_n \in \mathbb{R}^1.$$

Now we show that for sufficiently large ω the function

$$V(t, y) = \widetilde{y}^* H(t) \widetilde{y} + \omega y_n h(t) y_n$$

is the Lyapunov function, which for system (124) satisfies all conditions of the Lyapunov instability theorem for discrete systems.

Denote

$$K(t) = \widetilde{B}(t) e^{-d},$$
$$k(t) = \frac{|v_n(t+1)|}{|v_n(t)|} e^{-d}.$$

Then system (124) can be represented in the form

$$\begin{array}{ll} \widetilde{y}(t+1) = & K(t)\widetilde{y}(t) + q(t) y_n(t) + \widetilde{g}\big(t, \widetilde{y}(t), y_n(t)\big) \\ y_n(t+1) = & k(t) y_n(t) + g_n\big(t, \widetilde{y}(t), y_n(t)\big), \end{array} \tag{131}$$

where $q(t)$ is a certain bounded sequence, $\widetilde{g}$ and g_n are such that

$$g(t, y) = \begin{pmatrix} \widetilde{g}(t, y) \\ \\ g_n(t, y) \end{pmatrix}.$$

We introduce the following notations

$$V_1(t, y) = \widetilde{y}^* H(t) \widetilde{y},$$

$$V_2(t, y_n) = \omega h(t) y_n^2.$$

Then estimates (128), (130) give

$$\blacktriangle V_1(t, \widetilde{y}) \leq -\rho_1 |\widetilde{y}|^2 + q^2 H |y_n|^2 + \rho^2 (|\widetilde{y}| + |y_n|)^2 H +$$

$$+ 2H\big(Kq|\widetilde{y}||y_n| + K\rho(|\widetilde{y}| + |y_n|)|\widetilde{y}| + q\rho(|\widetilde{y}| + |y_n|)|y_n|\big),$$

$$\blacktriangle V_2(t, y_n) \leq -\omega\rho_3 |y_n|^2 + \omega h\big(\rho^2(|\widetilde{y}| + |y_n|)^2 + 2k\rho(|\widetilde{y}| + |y_n|)|y_n|\big),$$

where the positive values H,K,q,k,h are supremums for the corresponding norms. Since $A(t), H(t), h(t), q(t)$ are bounded for $t \in \mathbb{N}_0$, they are finite.

The above inequalities imply that for sufficiently large ω and sufficiently small ρ there exists a positive number θ such that

$$\blacktriangle V(t, \widetilde{y}, y_n) \leq -\theta\big(|\widetilde{y}|^2 + |y_n|^2\big). \tag{132}$$

From the boundedness of $H(t)$, $h(t)$ it follows that there exists a positive number a such that the inequality

$$|y|^2 \geq -aV(t, y), \qquad \forall y \in \mathbb{R}^n,$$

is valid for all t. Hence by (132) we have the inequality

$$\blacktriangle V(t, y) \leq a\theta V(t, y), \quad t = 0, 1..., \quad \forall y \in \mathbb{R}^n. \tag{133}$$

Suppose that the initial data $y(0)$ are such that $V(0, y(0)) < 0$. Then by (132) we have

$$V\big(t, y(t)\big) < 0, \quad \forall\, t = 0, 1...,$$

and by (122)

$$-V\big(t, y(t)\big) \geq (a\theta + 1)^t\big(-V\big(0, y(0)\big)\big).$$

This implies that by inequalities (129) and (130) the estimate

$$-\omega h(t) y_n(t)^2 \geq (a\theta + 1)^t\big(-V\big(0, y(0)\big)\big), \quad \forall t \geq 0$$

is valid. Then

$$y_n(t)^2 \geq \frac{(a\theta + 1)^t}{\omega \sup\limits_t \big(-h(t)\big)}\big(-V\big(0, y(0)\big)\big). \tag{134}$$

This inequality implies that the solution $y_n(t) \equiv 0$ is Lyapunov unstable.

Then the zero solution $y(t) \equiv 0$ is also Lyapunov unstable.

Since $d > 0$ and $U(t)$ is a unitary matrix, the solution $x(t) \equiv 0$ of the original system is Lyapunov unstable as well. ■

1.7.4 Instability criterion for the flow and cascade solutions

The problem arises naturally as to the weakening of instability conditions, which are due to Theorems 10 and 12. However the Perron effects impose restrictions on such weakening.

We now turn to continuous and discrete systems (64) and (64′), respectively.

Suppose, for a certain vector-function $\xi(t)$ the following relations

$$|\xi(t)| = 1, \quad \inf_{y \in \Omega} \big|X(t, y)\xi(t)\big| \geq \alpha(t), \quad \forall t \geq t_0 \tag{135}$$

hold.

Theorem 13 *[Leonov, 1998, Kuznetsov & Leonov, 2005[1]] Let for the function $\alpha(t)$ the following condition*

$$\lim_{t\to+\infty}\sup \alpha(t) = +\infty \tag{136}$$

be satisfied.

Then the flow (cascade) of solutions $x(t,y)$, $y \in \Omega$ is Lyapunov unstable.

Proof. Holding a certain pair $x_0 \in \Omega$ and $t \geq t_0$ fixed, we choose the vector y_0 in any δ-neighborhood of the point x_0 in such a way that

$$x_0 - y_0 = \delta\xi(t). \tag{137}$$

Let δ be so small that the ball of radius δ centered at x_0 is entirely placed in Ω. For any fixed values t, j and for the vectors x_0, y_0 there exists [Zorich, 1984] a vector $w_j \in \mathbb{R}^n$ such that

$$|x_0 - w_j| \leq |x_0 - y_0|,$$

$$x_j(t,x_0) - x_j(t,y_0) = X_j(t,w_j)(x_0 - y_0). \tag{138}$$

Here $x_j(t,x_0)$ is the jth component of the vector-function $x(t,x_0)$, $X_j(t,w)$ is the jth row of the matrix $X(t,w)$.

By (138) we have

$$|x(t,x_0) - x(t,y_0)| = \sqrt{\sum_j |X_j(t,w_j)(x_0-y_0)|^2} \geq$$

$$\geq \delta \max\{|X_1(t,w_1)\xi(t)|,\dots,|X_n(t,w_n)\xi(t)|\} \geq$$

$$\geq \delta \max_j \inf_\Omega |X_j(t,x_0)\xi(t)| = \delta \inf_\Omega \max_j |X_j(t,x_0)\xi(t)| \geq$$

$$\geq \frac{\delta}{\sqrt{n}} \inf_\Omega |X(t,x_0)\xi(t)| \geq \frac{\alpha(t)\delta}{\sqrt{n}}.$$

This estimate and conditions (136) imply that for any positive numbers ε and δ there exist a number $t \geq t_0$ and a vector y_0 such that

$$|x_0 - y_0| = \delta, \ \ |x(t,x_0) - x(t,y_0| > \varepsilon.$$

The latter means that the solution $x(t,x_0)$ is Lyapunov unstable. ■

Now we consider the hypotheses of Theorem 13.

The hypotheses of Theorem 13 is, in essence, the requirement that, at least, one Lyapunov exponent of the linearizations of the flow of solutions with the initial data from Ω is positive under the condition that the "unstable directions $\xi(t)$" (or unstable manifolds) of these solutions depend continuously on the initial data x_0. Actually, if this property holds, then, regarding (if necessary) the domain Ω as the union of the domains Ω_i, of arbitrary small diameter, on which conditions (135) and (136) are valid, we obtain the Lyapunov instability of the whole flow of solutions with the initial data from Ω.

Apply Theorem 13 to systems (33) and (33′).

For the solutions $x(t, t_0, x_0)$ with the initial data $t_0 = 0$,

$$x_1(0, x_{10}, x_{20}, x_{30}) = x_{10},$$

$$x_2(0, x_{10}, x_{20}, x_{30}) = x_{20},$$

$$x_3(0, x_{10}, x_{20}, x_{30}) = x_{30}$$

in the continuous case we have the following relations

$$x_1(t, x_{10}, x_{20}, x_{30}) = \exp(-at)x_{10},$$

$$\frac{\partial F(x,t)}{\partial x}|_{x=x(t,0,x_0)} = \begin{pmatrix} -a & 0 & 0 \\ 0 & -2a & 0 \\ -2\exp(-at)x_{10} & 1 & r(t) \end{pmatrix}, \tag{139}$$

where

$$r(t) = \sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a.$$

For discrete system we obtain

$$\frac{\partial F(x,t)}{\partial x}|_{x=x(t,0,x_0)} = \begin{pmatrix} \exp(-a) & 0 & 0 \\ 0 & \exp(-2a) & 0 \\ -2\exp(-at)x_{10} & 1 & r(t) \end{pmatrix}, \tag{139′}$$

where

$$r(t) = \frac{\exp\big((t+2)\sin\ln(t+2) - 2a(t+1)\big)}{\exp\big((t+1)\sin\ln(t+1) - 2at\big)}.$$

Solutions (65) and (65′) with matrices (139) and (139′), respectively, have the form

$$z_1(t) = \exp(-at)z_1(0),$$

$$z_2(t) = \exp(-2at)z_2(0), \tag{140}$$

$$z_3(t) = p(t)\big(z_3(0) + (z_2(0) - 2x_{10}z_1(0))\, q(t)\big).$$

Here in the continuous case we have

$$p(t) = \exp\big((t+1)\sin(\ln(t+1)) - 2at\big),$$

$$q(t) = \int_0^t \exp\big(-(\tau+1)\sin(\ln(\tau+1))\big)\, d\tau.$$

and in the discrete case

$$p(t) = \exp\big((t+1)\sin(\ln(t+1)) - 2at\big),$$

$$q(t) = \sum_{k=0}^{t-1} \exp(-(k+2)\sin\ln(k+2) + 2a).$$

Relations (140) give

$$X(t,0,x_0) = \begin{pmatrix} \exp(-at) & 0 & 0 \\ 0 & \exp(-2at) & 0 \\ -2x_{10}p(t)q(t) & p(t)q(t) & p(t) \end{pmatrix}.$$

If we assume that

$$\xi(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

then for $\Omega = \mathbb{R}^n$ and

$$\alpha(t) = \sqrt{\exp(-4at) + (p(t)q(t))^2}$$

relations (135) and (136) are satisfied (see estimate (28)).

Thus, by Theorem 13 any solution of system (33) is Lyapunov unstable.

Now we restrict ourselves to the consideration of the manifold

$$M = \{x_3 \in \mathbb{R}^1, \quad x_2 = x_1^2\}.$$

In this case the initial data of the unperturbed solution x_0 and the perturbed solution y_0 belong to the manifold M:

$$x_0 \in M, \quad y_0 \in M. \tag{141}$$

The analysis of the proof of Theorem 13 (see (137)) implies that the vector-function $\xi(t)$ satisfies the following additional condition: if (137) and (141) hold, then the inequality $\xi_2(t) \neq 0$ yields the relation $\xi_1(t) \neq 0$.

In this case (135) and (136) are not valid since for either $2x_{10}\xi_1(t) = \xi_2(t) \neq 0$ or $\xi_2(t) = 0$ the value

$$|X(t,x_0)|$$

is bounded on $[0,+\infty)$.

Thus, since in conditions (135) and (136) the uniformity with respect to x_0 is violated, for system (33) on the set M the Perron effects are possible under certain additional restrictions on the vector-function $\xi(t)$. ■

1.8 Conclusion

We summarize the investigations of stability by the first approximation.

Theorems 7 and 13 give a complete solution for the problem on the flows and cascade of solutions in the noncritical case when for small variations of the initial data of the original system the system of the first approximation preserves its stability (or instability in the certain "direction" $\xi(t)$).

Thus, the classical problem of stability by the first approximation of nonstationary motions is completely proved in the general case [Malkin, 1966].

The Perron effects are possible only on the boundaries of the flows that are either stable or unstable by the first approximation. From this point of view we have here the special case.

In the general case, the progress became possible since the theorem on finite increments permits us to reduce the estimate of the difference between perturbed and unperturbed solutions to the analysis of the first approximation system, linearized along a certain "third" solution of the original system. Such an approach makes the proof of the theorem "almost obvious".

Thus, the difficulties, arising in studying the individual solutions, are connected to the fact that these solutions can be situated on the boundaries of the flows that are stable (or unstable) by the first approximation. In this case a special situation occurs which requires the development of finer (and, naturally, more complicated) tools for investigation. Such methods of investigation of the individual solutions are given in the present study.

2 LYAPUNOV QUANTITIES AND LIMIT CYCLES

2.1 Introduction

The study of limit cycles of two-dimensional dynamical systems was stimulated by purely mathematical problems (Hilbert's sixteenth problem, the center-and-focus problem) as well as by many applied problems (the oscillations of electronic generators and electrical machines, the dynamics of populations, and the dangerous and safe boundaries of stability, see, for example, [Shilnikov *et al.,* 2001; Bautin & Leontovich, 1976; Andronov *et al.*, 1966; Blows & Perko, 1990; Perko, 1990; Anosov *et al.,* 1997] and others).

One of the central problems in studying small limit cycles in the neighborhood of equilibrium of two-dimensional dynamical systems is the computation of Lyapunov quantities [Poincare, 1885; Lyapunov, 1892; Cherkas, 1976; Marsden & McCracken, 1976; Lloyd, 1988; Yu, 1998; Yu & Han, 2005; Lynch, 2005; Roussarie, 1998; Reyn, 1994; Li, 2003; Chavarriga & Grau, 2003; Christopher & Li 2007; Yu & Chen, 2008]. The problems of greater dimension (when there are two purely imaginary roots and the rest are negative) can be reduced to two-dimensional problems with the help of procedure, proposed in [Lyapunov, 1892].

At present, there exist different methods for determining Lyapunov quantities and the computer realizations of these methods, which permit us to find Lyapunov quantities in the form of symbolic expressions, depending on expansion coefficients of the right-hand sides of equations of system (see., for example, surveys [Li, 2003] and others). These methods differ in complexity of algorithms and compactness of obtained symbolic expressions. The first method for finding Lyapunov quantities was suggested by Poincare [Poincare, 1885]. This method consists in sequential constructing time-independent holomorphic integral for approximations of the system. Further, different methods for computation, which use the reduction of system to normal forms, were developed (see, for example, [Li, 2003, Yu & Chen, 2008]).

Another approach to computation of Lyapunov quantities is related with finding approximations of solution of the system. So, in a classical approach [Lyapunov, 1892] it is used changes for reduction of "turn" time of all trajectories to a constant (as, for example, in the polar system of coordinates) and procedures for recurrent construction of solution approximations.

In the present work, together with the classical Poincare-Lyapunov method to calculate Lyapunov quantities based on constructing the time independent integral, a new method of computation of Lyapunov quantities is suggested based on constructing approximations of solution (as a finite sum in powers of degrees of initial data) in the original Euclidean system of coordinates and in the time domain. The advantages of this method are due to its ideological simplicity and visualization power. This approach can also be applied to the problem of distinguishing of isochronous center since it permits us to find out approximation of time of trajectory "turn" depend on initial data [Chavarriga & Grau, 2003, Gine, 2007, Gasull *et al.*, 1997].

While a general form of the first and second Lyapunov quantities were computed for two-dimensional real autonomous system (in terms of coefficients of the right-hand side of system) in the 40-50s of the last century [Bautin, 1949; Bautin, 1952; Serebryakova, 1959], the third Lyapunov quantity was computed only in certain special cases [Lloyd & Pearson, 1997; Yu & Han, 2005; Lynch, 2005].

In the present work, general formula for calculation of Lyapunov's third quantity is presented. The calculation of Lyapunov quantity by two different analytic methods involving modern software tools for symbolic computing enables us to justify the expression obtained for Lyapunov's third quantity.

The first steps in the development of this method were carried out in [Kuznetsov & Leonov 2007].

2.2 Calculation of Lyapunov quantities in the Euclidean system of coordinates and in the time domain

2.2.1 Approximation of solution in the Euclidean system of coordinates

Consider a system of two autonomous differential equations

$$\begin{aligned} \frac{dx}{dt} &= -y + f(x,y), \\ \frac{dy}{dt} &= x + g(x,y), \end{aligned} \tag{142}$$

where $x, y \in \mathbb{R}$ and the functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ have continuous partial derivatives of $(n+1)$th order in the open neighborhood U of radius R_U of the point $(x,y) = (0,0)$

$$f(\cdot,\cdot), g(\cdot,\cdot)\colon\ \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \in \mathbb{C}^{(n+1)}(U). \tag{143}$$

Suppose, the expansion of the functions f, g begins with the terms of at least the second order and therefore we have

$$f(0,0) = g(0,0) = 0, \quad \frac{df}{dx}(0,0) = \frac{df}{dy}(0,0) = \frac{dg}{dx}(0,0) = \frac{dg}{dy}(0,0) = 0. \tag{144}$$

Further we will use a smoothness of the functions f and g and will follow the first Lyapunov method on finite time interval (see f.e. classical works [Lindelof, 1894; Lefschetz, 1957; Cesari, 1959] and others). By assumption on smoothness (143) in the neighborhood U we have

$$\begin{aligned} f(x,y) &= \sum_{k+j=2}^{n} f_{kj}x^k y^j + o\big((|x|+|y|)^n\big) = f_n(x,y) + o\big((|x|+|y|)^n\big), \\ g(x,y) &= \sum_{k+j=2}^{n} g_{kj}x^k y^j + o\big((|x|+|y|)^n\big) = g_n(x,y) + o\big((|x|+|y|)^n\big). \end{aligned} \tag{145}$$

The existence condition of $(n+1)$ partial derivatives with respect to x and y for f and g is used for simplicity of exposition and can be weakened.

Let $x(t, x(0), y(0)),\ y(t, x(0), y(0))$ be a solution of system (142) with the initial data

$$x(0) = 0,\ y(0) = h. \tag{146}$$

Denote

$$x(t,h) = x(t,0,h),\ y(t,h) = y(t,0,h).$$

Below a time derivative will be denoted by x' and $\dot{x}$.

Lemma 12 *A positive number $H \in (0, R_U)$ exists such that for all $h \in [0, H]$ the solution $\big(x(t,h), y(t,h)\big)$ is defined for $t \in [0, 4\pi]$.*

The validity of lemma follows from condition (144) and the existence of two purely imaginary eigenvalues of the matrix of linear approximation of system (142).

This implies [Hartman, 1984] the following

Lemma 13 *If smoothness condition (143) is satisfied, then*

$$x(\cdot,\cdot), y(\cdot,\cdot) \in \mathbb{C}^{(n+1)}\big([0,4\pi] \times [0,H]\big) \tag{147}$$

Further we will consider the sufficiently small initial data $h \in [0, H]$, a finite time interval $t \in [0, 4\pi]$ and use a uniform boundedness of the solution $(x(t,h), y(t,h))$ and its mixed partial derivatives with respect to h and t up to the order $(n+1)$ inc in the set $[0, 4\pi] \times [0, H]$.

We apply now a well-known linearization procedure [Leonov & Kuznetsov, 2007]. From Lemma 13 it follows that for each fixed t the solution of system can be represented by the Taylor formula

$$\begin{aligned} x(t,h) &= h\frac{\partial x(t,\eta)}{\partial \eta}\Big|_{\eta=0} + \frac{h^2}{2}\frac{\partial^2 x(t,\eta)}{\partial \eta^2}\Big|_{\eta=h\theta_x(t,h)} \quad 0 \le \theta_x(t,h) \le 1, \\ y(t,h) &= h\frac{\partial y(t,\eta)}{\partial \eta}\Big|_{\eta=0} + \frac{h^2}{2}\frac{\partial^2 y(t,\eta)}{\partial \eta^2}\Big|_{\eta=h\theta_y(t,h)} \quad 0 \le \theta_y(t,h) \le 1. \end{aligned} \tag{148}$$

Note that by Lemma 13 and relation (148), the functions

$$\frac{h^2}{2}\frac{\partial^2 x(t,\eta)}{\partial\eta^2}|_{\eta=h\theta_x(t,h)},\ \frac{h^2}{2}\frac{\partial^2 y(t,\eta)}{\partial\eta^2}|_{\eta=h\theta_y(t,h)}$$

and their time derivatives are smooth functions of t and have the order of smallness $o(h)$ uniformly with respect to t on a considered finite time interval $[0, 4\pi]$.

Introduce the following denotations

$$\widetilde{x}_{h^k}(t) = \frac{\partial^k x(t,\eta)}{\partial^k\eta}|_{\eta=0},\ \widetilde{y}_{h^k}(t) = \frac{\partial^k y(t,\eta)}{\partial^k\eta}|_{\eta=0}.$$

We shall say that the sums

$$\begin{aligned} x_{h^m}(t,h) &= \sum_{k=1}^{m} \widetilde{x}_{h^k}(t)\frac{h^k}{k!} = \sum_{k=1}^{m} \frac{\partial^k x(t,\eta)}{\partial\eta^k}|_{\eta=0}\frac{h^k}{k!}, \\ y_{h^m}(t,h) &= \sum_{k=1}^{m} \widetilde{y}_{h^k}(t)\frac{h^k}{k!} = \sum_{k=1}^{m} \frac{\partial^k y(t,\eta)}{\partial\eta^k}|_{\eta=0}\frac{h^k}{k!} \end{aligned}$$

are the mth approximation of solution of system with respect to h. Substitute representation (148) in system (142). Then, equating the coefficients of h^1 and taking into account (144), we obtain

$$\begin{aligned} \frac{d\widetilde{x}_{h^1}(t)}{dt} &= -\widetilde{y}_{h^1}(t), \\ \frac{d\widetilde{y}_{h^1}(t)}{dt} &= \widetilde{x}_{h^1}(t). \end{aligned} \tag{149}$$

Hence, by conditions on initial data (146) for the first approximation with respect to h of the solution $(x(t,h), y(t,h))$, we have

$$x_{h^1}(t,h) = \widetilde{x}_{h^1}(t)h = -h\sin(t),\ y_{h^1}(t,h) = \widetilde{y}_{h^1}(t)h = h\cos(t). \tag{150}$$

Similarly, to obtain the second approximation $(x_{h^2}(t,h), y_{h^2}(t,h))$, we substitute representation

$$\begin{aligned} x(t,h) &= x_{h^2}(t,h) + \frac{h^3}{3!}\frac{\partial^3 x(t,\eta)}{\partial\eta^3}|_{\eta=h\theta_x(t,h)}, \\ y(t,h) &= y_{h^2}(t,h) + \frac{h^3}{3!}\frac{\partial^3 y(t,\eta)}{\partial\eta^3}|_{\eta=h\theta_y(t,h)}. \end{aligned} \tag{151}$$

in formula (145) for $f(x,y)$ and $g(x,y)$. Note that in expressions for f and g by virtue of (144) the coefficients of h^2 (denote them by $u^f_{h^2}$ and $u^g_{h^2}$, respectively) depend only on $\widetilde{x}_{h^1}(t)$ and $\widetilde{y}_{h^1}(t)$, i.e., by (150) they are known functions of time and are independent of the unknown functions $\widetilde{x}_{h^2}(t)$ and $\widetilde{y}_{h^2}(t)$. Thus, we have

$$\begin{aligned} f\big(x_{h^2}(t,h) + o(h^2), y_{h^2}(t,h) + o(h^2)\big) &= u^f_{h^2}(t)h^2 + o(h^2), \\ g\big(x_{h^2}(t,h) + o(h^2), y_{h^2}(t,h) + o(h^2)\big) &= u^g_{h^2}(t)h^2 + o(h^2). \end{aligned}$$

Substituting (151) in system (142), for the determination of $\widetilde{x}_{h^2}(t)$ and $\widetilde{y}_{h^2}(t)$ we obtain

$$\begin{aligned} \frac{d\widetilde{x}_{h^2}(t)}{dt} &= -\widetilde{y}_{h^2}(t) + u^f_{h^2}(t), \\ \frac{d\widetilde{y}_{h^2}(t)}{dt} &= \widetilde{x}_{h^2}(t) + u^g_{h^2}(t). \end{aligned} \tag{152}$$

Lemma 14 *For solutions of the system*

$$\begin{aligned} \frac{d\widetilde{x}_{h^k}(t)}{dt} &= -\widetilde{y}_{h^k}(t) + u^f_{h^k}(t), \\ \frac{d\widetilde{y}_{h^k}(t)}{dt} &= \widetilde{x}_{h^k}(t) + u^g_{h^k}(t) \end{aligned} \tag{153}$$

with the initial data

$$\widetilde{x}_{h^k}(0) = 0, \quad \widetilde{y}_{h^k}(0) = 0 \tag{154}$$

we have

$$\begin{aligned} &\widetilde{x}_{h^k}(t) = u^g_{h^k}(0)\cos(t) + \cos(t)\int_0^t \cos(\tau)\big((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau)\big)\,\mathrm{d}\tau + \\ &+ \sin(t)\int_0^t \sin(\tau)\big((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau)\big)\,\mathrm{d}\tau - u^g_{h^k}(t), \\ &\widetilde{y}_{h^k}(t) = u^g_{h^k}(0)\sin(t) + \sin(t)\int_0^t \cos(\tau)\big((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau)\big)\,\mathrm{d}\tau - \\ &- \cos(t)\int_0^t \sin(\tau)\big((u^g_{h^k}(\tau))' + u^f_{h^k}(\tau)\big)\,\mathrm{d}\tau. \end{aligned} \tag{155}$$

The relations (155) can be verified by direct differentiation.

Repeating this procedure for the determination of the coefficients $\widetilde{x}_{h^k}$ and $\widetilde{y}_{h^k}$ of the functions $u^f_{h^k}(t)$ and $u^g_{h^k}(t)$, by formula (155) we obtain sequentially the approximations $(x_{h^k}(t,h), y_{h^k}(t,h))$ for $k = 1, ..., n$. For $h \in [0, H]$ and $t \in [0, 4\pi]$ we have

$$\begin{aligned} x(t,h) &= x_{h^n}(t,h) + \frac{h^{n+1}}{(n+1)!}\frac{\partial^{n+1}x(t,\eta)}{\partial\eta^{n+1}}\Big|_{\eta=h\theta_x(t,h)} = \\ &= x_{h^n}(t,h) + o(h^n) = \sum_{k=1}^{n}\widetilde{x}_{h^k}(t)\frac{h^k}{k!} + o(h^n), \\ y(t,h) &= y_{h^n}(t,h) + \frac{h^{n+1}}{(n+1)!}\frac{\partial^{n+1}y(t,\eta)}{\partial\eta^{n+1}}\Big|_{\eta=h\theta_y(t,h)} = \\ &= y_{h^n}(t,h) + o(h^n) = \sum_{k=1}^{n}\widetilde{y}_{h^k}(t)\frac{h^k}{k!} + o(h^n), \\ &0 \le \theta_x(t,h) \le 1,\ 0 \le \theta_y(t,h) \le 1. \end{aligned} \tag{156}$$

Here by Lemma 13

$$\widetilde{x}_{h^k}(\cdot), \widetilde{y}_{h^k}(\cdot) \in \mathbb{C}^n([0, 4\pi]), \quad k = 1, ..., n \tag{157}$$

and the estimate $o(h^n)$ is uniform $\forall t \in [0, 4\pi]$. From (154) and by the choice of initial data in (149) we obtain

$$x_{h^k}(0,h) = x(0,h) = 0, \quad y_{h^k}(0,h) = y(0,h) = h, \qquad k = 1, ..., n.$$

2.2.2 Computation of Lyapunov quantities in the time domain

Consider for the initial datum $h \in (0, H]$ the time $T(h)$ of first crossing of the solution $\big(x(t,h), y(t,h)\big)$ of the half-line $\{x = 0,\ y > 0\}$. Complete a definition (by continuity) of the function $T(h)$ in zero: $T(0) = 2\pi$. Since by (150) the first approximation of solution crosses the half-line $\{x = 0, y > 0\}$ at the time 2π, then the crossing time can be represented as

$$T(h) = 2\pi + \Delta T(h),$$

where $\Delta T(h) = O(h)$. We shall say that $\Delta T(h)$ is a residual of crossing time.

By definition of $T(h)$ we have

$$x(T(h), h) = 0. \tag{158}$$

Since by (147), $x(\cdot,\cdot)$ has continuous partial derivatives with respect to either arguments up to the order n inc and $\dot{x}(t,h) = \cos(t)h + o(h)$, by the theorem on implicit function [Zorich, 2002], the function $T(\cdot)$ is n times differentiable. It is possible to show (for example, considering the function $z(t,h) = x(t,h)/h$ and completing its definition in zero by the function $x_{h^1}(t)$ or making use of special theorems of mathematical analysis) that $T(h)$ is also differentiable n times in zero. By the Taylor formula we have

$$T(h) = 2\pi + \sum_{k=1}^{n} \widetilde{T}_k h^k + o(h^n), \tag{159}$$

where $\widetilde{T}_k = \dfrac{1}{k!}\dfrac{d^k T(h)}{dh^k}$ (usually called period constants [Gine, 2007]). We shall say that the sum

$$\Delta T_k(h) = \sum_{j=1}^{k} \widetilde{T}_j h^j \tag{160}$$

is the kth approximation of the residual of the time $T(h)$ of the crossing of the solution $\big(x(t,h), y(t,h)\big)$ of the half-line $\{x = 0,\ y > 0\}$. Substituting relation (159) for $t = T(h)$ in the right-hand side of the first equation of (156) and denoting the coefficient of h^k by $\widetilde{x}_k$, we obtain the series $x(T(h), h)$ in terms of powers of h:

$$x(T(h), h) = \sum_{k=1}^{n} \widetilde{x}_k h^k + o(h^n). \tag{161}$$

In order to express the coefficients $\widetilde{x}_k$ by the coefficients $\widetilde{T}_k$ of the expansion of residual of crossing time we assume that in (156) $t = 2\pi + \tau$:

$$x(2\pi + \tau, h) = \sum_{k=1}^{n} \widetilde{x}_{h^k}(2\pi + \tau)\frac{h^k}{k!} + o(h^n). \tag{162}$$

By smoothness condition (157) we have

$$\widetilde{x}_{h^k}(2\pi + \tau) = \widetilde{x}_{h^k}(2\pi) + \sum_{m=1}^{n} \widetilde{x}^{(m)}_{h^k}(2\pi)\frac{\tau^m}{m!} + o(\tau^n), \quad k = 1, ..., n.$$

Substitute this representation in (162) for the solution $x(2\pi+\tau,h)$ for $\tau=\Delta T(h)$, and bring together the coefficients of the same exponents h. Since $(\Delta T(h))^n = O(h^n)$, by (158) and taking into account (159) for $T(h)$, we obtain

$$
\begin{aligned}
&h: 0=\widetilde{x}_1=\widetilde{x}_{h^1}(2\pi),\\
&h^2: 0=\widetilde{x}_2=\widetilde{x}_{h^2}(2\pi)+\widetilde{x}'_{h^1}(2\pi)\widetilde{T}_1,\\
&h^3: 0=\widetilde{x}_3=\widetilde{x}_{h^3}(2\pi)+\frac{1}{2}\widetilde{x}'_{h^2}(2\pi)\widetilde{T}_1+\widetilde{x}'_{h^1}(2\pi)\widetilde{T}_2+\frac{1}{2}\widetilde{x}''_{h^1}(2\pi)\widetilde{T}_1^2,\\
&\dots\\
&h^n: 0=\widetilde{x}_n=\dots
\end{aligned}
$$

From the above we sequentially find $\widetilde{T}_j$. The coefficients $T_{k=1,\dots,n-1}$ can be determined sequentially since the expression for $\widetilde{x}_k$ involves only the coefficients $T_{m<k}$ and the factor $\widetilde{x}'_{h^1}(2\pi)$ multiplying T_{k-1} is equal to -1.

We apply a similar procedure to determine the coefficients $\widetilde{y}_k$ of the expansion

$$y(T(h),h)=\sum_{k=1}^{n}\widetilde{y}_k h^k+o(h^n).$$

Substitute the representation

$$\widetilde{y}_{h^k}(2\pi+\Delta T(h))=\widetilde{y}_{h^k}(2\pi)+\sum_{m=1}^{n}\widetilde{y}^{(m)}_{h^k}(2\pi)\frac{\Delta T(h)^m}{m!}+o(h^n), \quad k=1,...,n$$

in the expression

$$y(2\pi+\Delta T(h),h)=\sum_{k=1}^{n}\widetilde{y}_{h^k}(2\pi+\Delta T(h))\frac{h^k}{k!}+o(h^n).$$

Equating the coefficients of the same exponents h, we obtain the following relations

$$
\begin{aligned}
&h: \quad \widetilde{y}_1=\widetilde{y}_{h^1}(2\pi),\\
&h^2: \quad \widetilde{y}_2=\widetilde{y}_{h^2}(2\pi)+\widetilde{y}'_{h^1}(2\pi)\widetilde{T}_1,\\
&h^3: \quad \widetilde{y}_3=\widetilde{y}_{h^3}(2\pi)+\frac{1}{2}\widetilde{y}'_{h^2}(2\pi)\widetilde{T}_1+\widetilde{y}'_{h^1}(2\pi)\widetilde{T}_2,+\frac{1}{2}\widetilde{y}''_{h^1}(2\pi)\widetilde{T}_1^2,\\
&\dots\\
&h^n: \quad \widetilde{y}_n=\dots
\end{aligned}
$$

for the sequential determination of $\widetilde{y}_{i=1,\dots,n}$. Here $\widetilde{y}_{h^{k=1,..,n}}(\cdot)$ and $\widetilde{T}_{k=1,..,n-1}$ are the obtained above quantities.

Thus, for $n=2m+1$ under the condition $f(\cdot,\cdot),g(\cdot,\cdot)\in\mathbb{C}^{(2m+2)}(U)$ we sequentially obtained the approximations of the solution $\big(x(t,h),y(t,h)\big)$ at the time $t=T(h)$ of the first crossing of the half-line $\{x=0, y>0\}$ accurate to $o(h^{2m+1})$ and the approximation of the time $T(h)$ itself accurate to $o(h^{2m})$. If in this case $\widetilde{y}_k=0$ for $k=2,..,2m$, then $\widetilde{y}_{2m+1}$ is called the mth Lyapunov quantity L_m. Note, that, according to the Lyapunov theorem, the first nonzero coefficient of the expansion $\widetilde{y}_i$ is always of an odd number and for sufficiently small initial data h the sign of $\widetilde{y}_i$ (of the Lyapunov quantity) designates a qualitative behavior (winding or unwinding) of the trajectory $\big(x(t,h), y(t,h)\big)$ on plane [Lyapunov, 1892].

Expression for the first, second, and third Lyapunov quantities in the general form can be found in Appendix 1.

2.2.3 Duffing equation

Consider the Duffing equation $\ddot{x} + x + x^3 = 0$ written as the system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + x^3. \end{aligned} \tag{163}$$

Suppose that $x_0 = 0, y_0 = h_y$. Then

$$y(t)^2 + x(t)^2 + \frac{1}{2}x(t)^4 = y_0^2, \tag{164}$$

and, therefore,

$$y(t) = \pm\sqrt{h_y^2 - x(t)^2 - \frac{x(t)^4}{2}}, \quad x(t)^2 = -1 + \sqrt{1 + 2h_y^2 - 2y(t)^2}.$$

By virtue of system (163), this gives us

$$\frac{dt}{dy} = \frac{1}{x(1+x^2)} = \frac{1}{\sqrt{-1+\sqrt{1+2h_y^2-2y^2}}\sqrt{1+2h_y^2-2y^2}}.$$

Then, for the intersection time $T(h_y)$, we have

$$T(h_y) = 4\int\limits_0^{h_y} \frac{\mathrm{d}y}{\sqrt{-1+\sqrt{1+2h_y^2-2y^2}}\sqrt{1+2h_y^2-2y^2}}.$$

Having changed the variables

$$y = h_y\cos(z), z = \arccos\frac{y}{h_y}, y = h_y \Rightarrow z = \frac{\pi}{2}, \mathrm{d}y = -h_y\sin(z)\mathrm{d}z$$

we obtain

$$T(h_y) = \int\limits_0^{\pi/2} \frac{-h_y\sin(z)\mathrm{d}z}{\sqrt{-1+\sqrt{1+2h_y^2\sin^2 z}}\sqrt{1+2h_y^2\sin^2 z}}.$$

Expanding $T(h))$ in powers of h gives an estimate for the time when the trajectory meets the vertical half-line $(x = 0, y > 0)$

$$T(h_y) = 2\pi - \frac{3\pi}{4}h_y^2 + \frac{105\pi}{128}h_y^4 - \frac{1155\pi}{1024}h_y^6 + o(h_y^6),$$

this corresponds to the values obtained by the above algorithm. Below, we list the solutions obtained by the solutions approximation algorithm:

$$\widetilde{x}_{h^1}(t) = -\sin(t), \quad \widetilde{y}_{h^1}(t) = \cos(t);$$

$$\widetilde{x}_{h^2}(t) = \widetilde{y}_{h^2}(t) = 0;$$

$$\widetilde{x}_{h^3}(t) = \frac{1}{8}\cos(t)^2\sin(t) - \frac{3}{8}t\cos(t) + \frac{1}{4}\sin(t),$$
$$\widetilde{y}_{h^3}(t) = -\frac{3}{8}t\sin(t) + \frac{3}{8}\cos(t) - \frac{3}{8}\cos(t)^3;$$

$$\widetilde{x}_{h^4}(t) = \widetilde{y}_{h^4}(t) = 0;$$

$$\begin{aligned}\widetilde{x}_{h^5}(t) &= -\frac{1}{64}\sin(t)\cos(t)^4 - \frac{45}{256}\cos(t)^2\sin(t) + \frac{69}{256}\cos(t)t + \\ &+ \frac{9}{128}\sin(t)t^2 - \frac{7}{32}\sin(t) + \frac{9}{64}t\cos(t)^3, \\ \widetilde{y}_{h^5}(t) &= \frac{33}{256}\sin(t)t + \frac{5}{64}\cos(t)^5 + \frac{27}{64}t\cos(t)^2\sin(t) - \\ &- \frac{9}{128}\cos(t)t^2 + \frac{83}{256}\cos(t)^3 - \frac{103}{256}\cos(t),\end{aligned}$$

Thus, a periodic solution is approximated by a series with nonperiodic coefficients.

Note that Lyapunov quantities computed above are equal to zero,

$$\mathrm{L}_1 = \mathrm{L}_2 = ... = 0,$$

which agrees with condition (164).

2.2.4 Application of Lyapunov function in order to weaken smoothness requirements when calculating Lyapunov quantities

Suppose that $n = 2m$ and

$$f(\cdot,\cdot), g(\cdot,\cdot) \in \mathbb{C}^{(2m+1)}(U). \tag{165}$$

In this case, the above procedure enables us to calculate only the coefficients of $\widetilde{y}_1, ..., \widetilde{y}_{2m}$ and does not work for $\widetilde{y}_{2m+1}$ (to calculate the latter, we formally need that $f(\cdot,\cdot), g(\cdot,\cdot) \in \mathbb{C}^{(2m+2)}(U)$).

In the case when $\widetilde{y}_{k=2,...,2m} = 0$, with the scope to estimate the qualitative behavior of the trajectories in a neighborhood of zero, let us consider a Lyapunov function and its derivative by virtue of system (142)

$$V(x,y) = \frac{(x^2+y^2)}{2}, \quad \dot{V}(x,y) = xf(x,y) + yg(x,y). \tag{166}$$

Introduce the notation:

$$\begin{aligned}L(h) &= \int_0^{T(h)} \dot{V}\big(x(t,h), y(t,h)\big)\,\mathrm{d}t \\ &= V\big(x(T(h),h), y(T(h),h)\big) - V\big(x(0,h), y(0,h)\big).\end{aligned}$$

Lemma 15

$$
\begin{aligned}
L(h) \;=\; & \int\limits_{0}^{2\pi+\Delta T_n(h)} x_{h^n}(t,h) f\big(x_{h^n}(t,h), y_{h^n}(t,h)\big) + \\
& + y_{h^n}(t,h) g\big(x_{h^n}(t,h), y_{h^n}(t,h)\big)\, \mathrm{d}t + o(h^{n+2}).
\end{aligned} \tag{167}
$$

Proof. Applying (156) gives

$$
\begin{aligned}
x(t,h) = x_{h^n}(t,h) + o(h^n) = \sum_{k=1}^{n} \widetilde{x}_{h^k}(t) \frac{h^k}{k!} + o(h^n), \\
y(t,h) = y_{h^n}(t,h) + o(h^n) = \sum_{k=1}^{n} \widetilde{y}_{h^k}(t) \frac{h^k}{k!} + o(h^n).
\end{aligned}
$$

Here, coefficients $\widetilde{x}_{h^k}(t)$ are bounded functions of time, and the estimate $o(h^n)$ is uniform for any $\forall t \in [0, 4\pi]$. From (166) and (144), we have

$$
\dot{V}(x(t,h), y(t,h)) = x(t,h) f\big(x(t,h), y(t,h)\big) + y(t,h) g\big(x(t,h), y(t,h)\big) = o(h^2).
$$

Therefore,

$$
\int\limits_{\Delta T_n(h)}^{\Delta T_n(h)+o(h^n)} \dot{V}(x(t,h), y(t,h))\, \mathrm{d}t = o(h^{n+2}). \tag{168}
$$

Allowing for (145) and (166), we obtain the representation

$$
\dot{V}(x(t,h), y(t,h)) = \dot{V}(x_{h^n}(t,h), y_{h^n}(t,h)) + o(h^{n+2}),
$$

the estimate $o(h^{n+2})$ being uniform for any $\forall t \in [0, 4\pi]$. On account of (168), this gives us

$$
L(h) = \int\limits_{0}^{2\pi+T_n(h)} \dot{V}(x_{h^n}(t,h), y_{h^n}(t,h))\, \mathrm{d}t + o(h^{n+2}).
$$

■

Inserting the solutions in the form (167) into expression (156) for $L(h)$, integrating, and grouping together the coefficients of the same powers of h, we obtain

$$
L(h) = \sum_{k=3}^{2m+2} \widetilde{L}_k h^k + o(h^{2m+2}).
$$

Lemma 16 *Suppose that system (142) is sufficiently smooth,*

$$
f(\cdot,\cdot), g(\cdot,\cdot) \in \mathbb{C}^{(2m+2)}(U)
$$

and

$$
\widetilde{y}_k = 0 \quad k = 2, .., 2m, \quad \widetilde{y}_{2m+1} = \mathrm{L}_m \neq 0.
$$

Then $\widetilde{L}_{2m+2} = \mathrm{L}_m$.

Proof. From the conditions of the Lemma 16 and definition of $T(h)$, we have

$$x(T(h),h) = 0, \quad y(T(h),h) = h + \widetilde{y}_{2m+1}h^{2m+1}.$$

Then, applying (166) gives

$$\begin{aligned} V\big(x(T(h),h), y(T(h),h)\big) &= \frac{y(T(h),h)^2}{2} = \frac{h^2}{2} + y_{2m+1}h^{2m+2} + o(h^{2m+2}), \\ V\big(x(0,h), y(0,h)\big) &= \frac{h^2}{2}. \end{aligned}$$

Therefore,

$$L(h) = y_{2m+1}h^{2m+2} + o(h^{2m+2}).$$

■

Note that if we weaken the smoothness requirement,

$$f(\cdot,\cdot), g(\cdot,\cdot) \in \mathbb{C}^{(2m)}(U), \tag{169}$$

and require that the estimate $o(h^n)$ and the estimate (156) be uniform in t, then the above procedure to calculate $\widetilde{L}_{2m+2}$ enables us to extend the notion of Lyapunov quantity L_m to insufficiently smooth systems (if $\widetilde{L}_3 = ... = \widetilde{L}_{2m+1} = 0$, then the sign of $\widetilde{L}_{2m+2}$ also determines the qualitative behavior of the trajectory for sufficiently small initial data).

2.3 Classical method for computation of Lyapunov quantities

Following the classical work [Poincare, 1885; Lyapunov, 1892], we consider a problem of computation of Lyapunov quantities by constructing the time independent integral $V(x,y)$ for system (142). Since $V_2(x,y) = \dfrac{(x^2+y^2)}{2}$ is an integral of system of the first approximation and for the right-hand side of system smoothness condition (145) is satisfied, then in certain small neighborhood of zero state we seek the approximation of integral in the form

$$V(x,y) = \frac{x^2+y^2}{2} + V_3(x,y) + ... + V_{n+1}(x,y) \tag{170}$$

Here $V_k(x,y)$ are the following homogeneous polynomials

$$V_k(x,y) = \sum_{i+j=k} V_{i,j}x^iy^j \quad k = 3,...,n+1$$

with the unknown coefficients $\{V_{i,j}\}_{i+j=k,\, i,j\geqslant 0}$. By (145) for the derivative of $V(x,y)$ by virtue of system (142) we have

$$\dot{V}(x,y) = \frac{\partial V(x,y)}{\partial x}(-y + \sum_{k+j=2}^{n} f_{kj}x^ky^j) + \frac{\partial V(x,y)}{\partial y}(x + \sum_{k+j=2}^{n} g_{kj}x^ky^j) + o\big((|x|+|y|)^{n+1}\big). \tag{171}$$

The coefficients of the forms V_k can always be chosen in such a way that

$$\dot{V}(x,y) = w_1(x^2+y^2)^2 + w_2(x^2+y^2)^3 + ... + o\big((|x|+|y|)^{n+1}\big). \qquad (172)$$

Here w_i are expressions depending only on coefficients of the functions f and g.

Then sequentially determining the coefficients of the forms V_k for $k = 3, ...$ (for that at each step it is necessary to solve a system of $(k+1)$ linear equations), from (171) and (172) we obtain the coefficient w_m that is the first not equal to zero

$$\dot{V}(x,y) = w_m(x^2+y^2)^{m+1} + o\big((|x|+|y|)^{2m+2}\big).$$

The expression w_m is usually [Chavarriga & Grau, 2003] called a Poincare-Lyapunov constant ($2\pi w_m$ is mth Lyapunov quantity [Frommer, 1934]). Let the additional conditions [Lynch, 2005]

$$V_{2m,2m+2} + V_{2m+2,2m} = 0, \quad V_{2m,2m} = 0$$

be satisfied. Then at the kth step of iteration the coefficients $\{V_{i,j}\}_{i+j=k}$ can be determined uniquely from the linear equations system via the coefficients $\{f_{ij}\}_{i+j<k}$ and $\{g_{ij}\}_{i+j<k}$ and the coefficients $\{V_{i,j}\}_{i+j<k}$, determined at the previous steps of iteration.

2.4 The Lienard equation

Assuming in (142)

$$f(x,y) \equiv 0, \ \frac{dg(x,y)}{dy} = g_{x1}(x), \ g(x,0) = g_{x0}(x), \ \frac{dg_{x0}}{dx}(0) = 0,$$

we obtain the following system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + g_{x1}(x)y + g_{x0}(x), \end{aligned} \qquad (173)$$

or the equivalent Lienard equation

$$\ddot{x} + x + \dot{x}g_{x1}(x) + g_{x0}(x) = 0.$$

Let $g_{x1}(x) = g_{11}x + ...,\ g_{x0}(x) = g_{11}x^2 + ...$ Then

$$\mathrm{L}_1 = -\frac{\pi}{4}(g_{20}g_{11} - g_{21}).$$

If $g_{21} = g_{20}g_{11}$, then $\mathrm{L}_1 = 0$ and

$$\mathrm{L}_2 = \frac{\pi}{24}(3g_{41} - 5g_{20}g_{31} - 3g_{40}g_{11} + 5g_{20}g_{30}g_{11}).$$

If $g_{41} = \frac{5}{3}g_{20}g_{31} + g_{40}g_{11} - \frac{5}{3}g_{20}g_{30}g_{11}$, then $\mathrm{L}_2 = 0$ and

$$\begin{aligned} \mathrm{L}_3 = \ & -\frac{\pi}{576}(70g_{20}^3g_{30}g_{11} + 105g_{20}g_{51} + 105g_{30}^2g_{11}g_{20} + 63g_{40}g_{31} \\ & -63g_{11}g_{40}g_{30} - 105g_{30}g_{31}g_{20} - 70g_{20}^3g_{31} - 45g_{61} - 105g_{50}g_{11}g_{20} + 45g_{60}g_{11}). \end{aligned}$$

The above expressions can be obtained by MatLab programm in Appendix 2 and for L_4 and L_5 see [Leonov & Kuznetsova, 2008].

2.5 Application of Lyapunov quantities for investigation of quadratic systems

Let us consider transformation of quadratic system to a special type of Lienard system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -F(x)y - G(x), \end{aligned} \tag{174}$$

where

$$\begin{aligned} F(x) &= (Ax+B)x|x+1|^{q-2}, \\ G(x) &= (C_1x^3 + C_2x^2 + C_3x + 1)x\frac{|x+1|^{2q}}{(x+1)^3}. \end{aligned} \tag{175}$$

We have the following results [Leonov, 1997; Leonov *et al.* 2008].

Lemma 17 *Suppose, for the coefficients A, B, C_1, C_2, C_3, q of equation (174) the relations*

$$\frac{(B-A)}{(2q-1)^2}\left((1-q)B + (3q-2)A\right) = 2C_2 - 3C_1 - C_3, \tag{176}$$

$$\frac{(B-A)}{(2q-1)^2}\left(B + 2(q-1)A\right) = C_2 - 2C_1 - 1. \tag{177}$$

are satisfied. Then equation (174) can be reduced to the quadratic system

$$\begin{aligned} \dot{x} &= p(x,y) = a_1x^2 + b_1xy + \alpha_1x + \beta_1y, \\ \dot{y} &= q(x,y) = a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y. \end{aligned} \tag{178}$$

with the coefficients $b_1 = 1$, $\alpha_1 = 1$, $\beta_1 = 1$, $c_2 = -q$, $\alpha_2 = -2, \beta_2 = -1$,

$$\begin{aligned} a_1 &= 1 + \frac{B-A}{2q-1}, \\ a_2 &= -(q+1)a_1^2 - Aa_1 - C_1, \\ b_2 &= -A - a_1(2q+1). \end{aligned} \tag{179}$$

Then by the above relations for the Lyapunov quantities L_1 and L_2, we obtain the following

Lemma 18 *if $L_1 = L_2 = 0$, $5A - 2Bq - 4B = 0$ and $A \neq B,\ AB \neq 0,\ q \neq \frac{1}{2}$ then*

$$C_1 = (q+3)\frac{B^2}{25} - \frac{(1+3q)}{5},$$

$$C_2 = \left(15(1-2q) + 3B^2\right)\frac{1}{25},$$

$$C_3 = \frac{3(3-q)}{5};$$

$$L_3 = -\frac{\pi B(q+2)(3q+1)[5(q+1)(2q-1)^2 + B^2(q-3)]}{20000}.$$

Thus, if the conditions of Lemma 18 and $L_3 \neq 0$, then by small disturbances of system we can obtain three "small" cycles around the zero equilibrium of system and seek "large" cycles on a plane of the rest two coefficients (B,q).

Lemma 19 *For $b_1 \neq 0$ system (178) can be reduced to the Lienard equation (174) with the functions*

$$F(x) = R(x)e^{p(x)} = R(x)|\beta_1 + b_1 x|^q,$$
$$G(x) = P(x)e^{2p(x)} = P(x)|\beta_1 + b_1 x|^{2q}.$$

Here $q = -\dfrac{c_2}{b_1}$,

$$R(x) = -\frac{(b_1b_2 - 2a_1c_2 + a_1b_1)x^2 + (b_2\beta_1 + b_1\beta_2 - 2\alpha_1c_2 + 2a_1\beta_1)x + \alpha_1\beta_1 + \beta_1\beta_2}{(\beta_1 + b_1x)^2},$$

$$P(x) = -\left(\frac{a_2x^2 + \alpha_2x}{\beta_1 + b_1x} - \frac{(b_2x + \beta_2)(a_1x^2 + \alpha_1x)}{(\beta_1 + b_1x)^2} + \frac{c_2(a_1x^2 + \alpha_1x)^2}{(\beta_1 + b_1x)^3}\right).$$

The above results were applied to quadratic systems and the experiments for computing "large" cycles were performed. Such computer experiments were carried out by Kudryashova [Leonov *et al.* 2008]. In these experiments the reduction of quadratic system to the Lienard equation of special form (174)-(175) was used and with its help a set of parameters B, q (Fig. 5), which correspond to the existence of "large" cycle, was estimated.

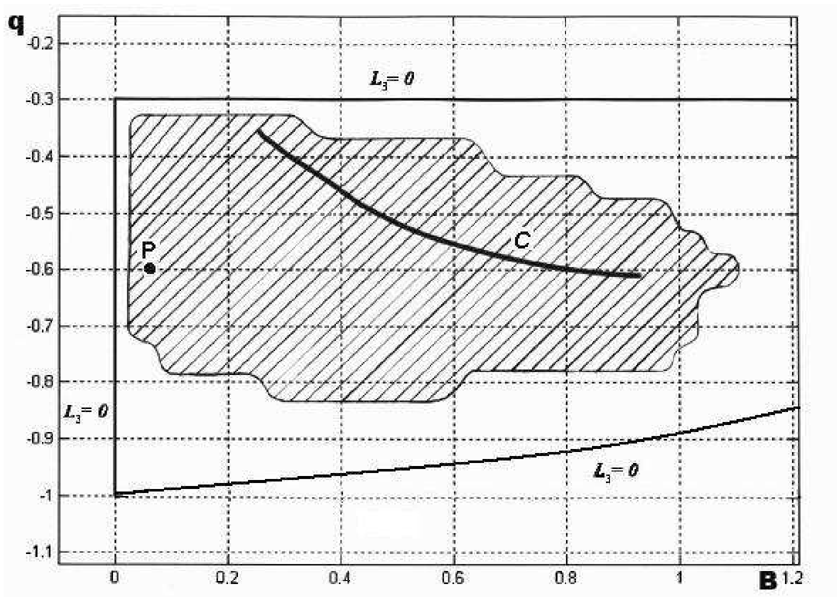

FIGURE 5 Domain of existence of "large" limit cycles

In Fig. 5 it is shown a domain bounded by lines, which correspond to the lines of reversal sign of the third Lyapunov quantity. The curve C in the graph is a curve of the parameters B and q of the Lienard system, which correspond to parameters of quadratic system, such that for these parameters the results on the existence of four cycles were obtained in [Shi, 1980].

Since two Lyapunov quantities are equal to zero, by small disturbances it is possible to construct systems with four cycles for the considered domain of parameters: three small cycles around one equilibrium and one large cycle around another equilibrium.

Note that if the conditions of Lemma 18 are satisfied, then the changes of the time $t \to -t$ and the parameter of system $B \to -B$ don't modify system (175). Therefore, analogous domain, of existence of large cycle, which is symmetric about the straight line $B = 0$, holds.

These results were applied to quadratic systems and the experiments for computing "large" cycles were performed. Our experience of computations shows that it is practically impossible to trace "small" cycles in the neighborhood of equilibrium, where the zero and first Lyapunov quantities are equal to zero. However in a number of computer experiments we can distinctly see "large" cycles.

For example in Fig. 6 it is shown a "large" cycle for the system

$$\begin{aligned} \dot{x} &= 0.99x^2 + xy + x + y, \\ \dot{y} &= -0.58x^2 + 0.17xy + 0.6y^2 - 2x - y, \end{aligned}$$

the parameters of which correspond to the point P in Fig. 5.

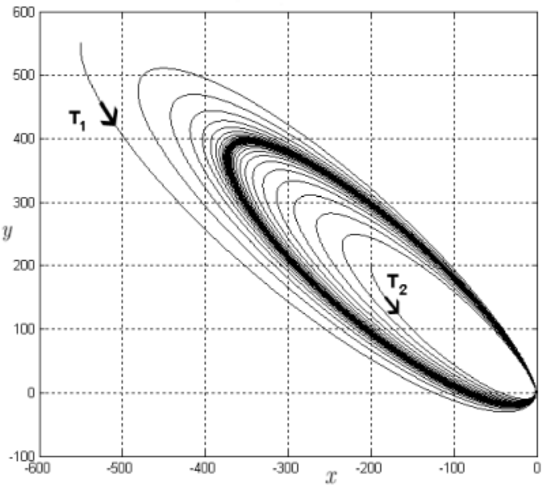

FIGURE 6 Stable limit cycle in quadratic system

3 STABILITY AND OSCILLATIONS IN PENDULUM-LIKE SYSTEMS

3.1 Synchronization of two metronomes

3.1.1 Introduction

The synchronization phenomenon of two pendulum clocks was first discovered by Christian Huygens in 1662. He observed what is now called the anti-phase synchronization of two pendulums of the clocks attached to a common support beam. Regardless of the initial conditions these two pendulums converged after some transient process to an oscillatory regime characterized by identical frequency of the oscillations, while the two pendulum angles moved in anti-phase. Huygens [Huygens, 1669,1986] found an explanation of this phenomenon noticing that imperfect synchronization resulted in small beam oscillation that in turn drove the pendulum towards the agreement. Though his explanation is physically correct, rigorous analytical results become available later on with invention of differential calculus. Within 300 years of Huygens' discovery it turned out that this phenomenon finds a lot of potential applications in different fields of science and engineering. For some related analytical results, see e.g.[Bennet *et. al.*, 2005, Pogromsky *et. al.*, 2005].

Together with anti-phase oscillations, a similar setup with two metronomes on a common support demonstrates also in-phase synchronization, where metronomes' pendulum agree not only in frequency but also in angles [Oud *et. al.*, 2006].

In the book [Blekhman, 1988], Blekhman also discusses Huygens' observations, and recounts the results of a laboratory reproduction of the coupled clocks as well as presenting a theoretical analysis of oscillators coupled through a common supporting frame. He predicted that both in-phase and anti-phase motions are stable under the same circumstances.

The problem of analytical study of in-phase synchronization turns out to be more difficult. In this work this problem is considered for the model of two metronomes on the common support proposed in [Pantaleone, 2002].

3.1.2 Problem statement

We consider a system consisting of two metronomes resting on a light wooden board that sits on two empty soda cans

FIGURE 7 [Pantaleone, 2002] Two metronomes sitting on a light wooden board which lies on two empty soda cans

The motion of such system can be described [Pantaleone, 2002] by the following equations

$$\begin{aligned} &m(l\ddot{\phi}_i + \ddot{x}\cos\phi_i) + mg\sin\phi_i = \kappa^{esc}\big(1 - \frac{\widehat{\phi_i}^2}{\Phi^2}\big)\dot{\phi}_i = f_{esc}(\phi_i)\dot{\phi}_i \\ &M\ddot{x} + m\sum_{i=1}^{2}\big(l\ddot{\phi}_i\cos\phi_i - l\dot{\phi}_i^{\,2}\sin\phi_i + \ddot{x}\big) = 0 \end{aligned} \tag{180}$$

Here m is a mass of each weight of metronomes, M is a mass of platform, ϕ_i is an angle of deviation of the ith pendulum of metronome from a vertical, l is a length of the pendulum of metronome, $f_{esc}(\cdot)$ is an internal force of metronome, (κ^{esc} is a small parameter, $\widehat{\phi_i} = \phi_i \mod 2\pi$), g is a gravitational acceleration, and x is a horizontal displacement of platform beginning from equilibrium.

We find conditions under which the in-phase regime occurs.

To this end, we need new variables

$$\theta_+ = \frac{\phi_1 + \phi_2}{2}, \quad \theta_- = \frac{\phi_1 - \phi_2}{2}.$$

In this case by trigonometric formulas we obtain

$$\begin{aligned} &\frac{\sin\phi_1 + \sin\phi_2}{2} = \sin\theta_+\cos\theta_-, \frac{\sin\phi_1 - \sin\phi_2}{2} = \sin\theta_-\cos\theta_+, \\ &\frac{\cos\phi_1 + \cos\phi_2}{2} = \cos\theta_+\cos\theta_-, \frac{\cos\phi_1 - \cos\phi_2}{2} = \sin\theta_-\sin\theta_+. \end{aligned}$$

The second equation of (180) gives the following expression for the acceleration $\ddot{x}$ of the platform

$$\ddot{x} = -\frac{ml\big((\sin\phi_1)'' + (\sin\phi_2)''\big)}{(M + 2m)}.$$

This implies that, written a half-sum and half-difference of equations for the motion of weights of metronomes (180) in new variables θ_+, θ_-, we obtain

$$\begin{aligned} & m(l\ddot{\theta_+} - \frac{2ml(\sin\theta_+ \cos\theta_-)''}{M+2m}\cos\theta_+ \cos\theta_- + g\sin\theta_+ \cos\theta_-) = \\ & = (f_{esc}(\phi_1) + f_{esc}(\phi_2))/2, \\ & m(l\ddot{\theta_-} - \frac{2ml(\sin\theta_+ \cos\theta_-)''}{M+2m}\sin\theta_- \sin\theta_+ + g\sin\theta_- \cos\theta_+) = \\ & = (f_{esc}(\phi_1) - f_{esc}(\phi_2))/2, \end{aligned} \tag{181}$$

$$\begin{aligned} & f_{esc}(\phi_1)\dot{\phi_1} + f_{esc}(\phi_2)\dot{\phi_2} = \kappa^{esc}(1 - \frac{\phi_1^2}{\Phi^2})\dot{\phi}_1 + \\ & + \kappa^{esc}(1 - \frac{\phi_2^2}{\Phi^2})\dot{\phi}_2 = \\ & = \kappa^{esc}(2\dot{\theta_+} - \frac{1}{\Phi^2}((\theta_+ + \theta_-)^2(\dot{\theta_+} + \dot{\theta_-}) + (\theta_+ - \theta_-)^2(\dot{\theta_+} - \dot{\theta_-}))) = \\ & = \kappa^{esc}(2\dot{\theta_+} - \frac{1}{\Phi^2}(2\theta_+^2\dot{\theta_+} + 4\theta_+\theta_-\dot{\theta_-} + 2\theta_-^2\dot{\theta_+})), \end{aligned}$$

$$\begin{aligned} & f_{esc}(\phi_1)\dot{\phi_1} - f_{esc}(\phi_2)\dot{\phi_2} = \kappa^{esc}(1 - \frac{\phi_1^2}{\Phi^2})\dot{\phi}_1 - \kappa^{esc}(1 - \frac{\phi_2^2}{\Phi^2})\dot{\phi}_2 = \\ & = \kappa^{esc}(2\dot{\theta_-} - \frac{1}{\Phi^2}((\theta_+ + \theta_-)^2(\dot{\theta_+} + \dot{\theta_-}) - (\theta_+ - \theta_-)^2(\dot{\theta_+} - \dot{\theta_-}))) = \\ & = \kappa^{esc}(2\dot{\theta_-} - \frac{1}{\Phi^2}(2\theta_+^2\dot{\theta_-} + 4\theta_+\theta_-\dot{\theta_+} + 2\theta_-^2\dot{\theta_-})). \end{aligned} \tag{182}$$

It follows that in the mechanical system there can occur the in-phase regime ($2\theta_- = \phi_1 - \phi_2 = 0 \Rightarrow \phi_1 \equiv \phi_2$), in which for the half-sum of angles of deviation of metronomes pendulum $\theta = \theta_+$ satisfies the equation

$$m(l\ddot{\theta} - \frac{2ml(\sin\theta)''}{M+2m}\cos\theta + g\sin\theta) = \kappa^{esc}(1 - \frac{\theta^2}{\Phi^2})\dot{\theta}. \tag{183}$$

Having performed the transformations

$$m(l\ddot{\theta} - \frac{2ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)}{M+2m}\cos\theta + g\sin\theta) = \kappa^{esc}(1 - \frac{\theta^2}{\Phi^2})\dot{\theta}, \tag{184}$$

$$\ddot{\theta}(1 - \frac{2m\cos^2\theta}{M+2m}) + \dot{\theta}^2\frac{2m\sin\theta\cos\theta}{M+2m} + \frac{g}{l}\sin\theta = \frac{\kappa^{esc}}{ml}(1 - \frac{\theta^2}{\Phi^2})\dot{\theta}, \tag{185}$$

$$\ddot{\theta}\frac{M+2m\sin^2\theta}{M+2m} + \dot{\theta}^2\frac{m\sin 2\theta}{M+2m} + \frac{g}{l}\sin\theta = \frac{\kappa^{esc}}{ml}(1 - \frac{\theta^2}{\Phi^2})\dot{\theta}, \tag{186}$$

for

$$\varepsilon_{m,M}(\theta) = \frac{2m\cos^2\theta}{M+2m\sin^2\theta},$$

$$\kappa_{ml}^{esc} = \frac{\kappa^{esc}}{ml}, \quad \varepsilon_{\frac{2m}{M}} = \frac{2m}{M},$$

we obtain

$$\ddot{\theta} - \dot{\theta}F(\theta, \kappa^{esc}) + \dot{\theta}^2 H(\theta) + G(\theta) = 0, \tag{187}$$

$$F(\theta,\kappa^{esc}) = \kappa^{esc}\frac{M+2m}{ml(M+2m\sin^2\theta)}(1-\frac{\theta^2}{\Phi^2}) =$$
$$= \kappa^{esc}_{ml}(1-\frac{\theta^2}{\Phi^2})(1+\varepsilon_{m,M}(\theta)),$$
$$H(\theta) = \frac{m\sin 2\theta}{M+2m\sin^2\theta} = \tan\theta\varepsilon_{m,M}(\theta),$$
$$G(\theta) = \frac{g(M+2m)\sin\theta}{l(M+2m\sin^2\theta)} = \frac{g\sin\theta}{l}(1+\varepsilon_{m,M}(\theta))$$

Then for linearization at the equilibrium $\theta = 0, \dot{\theta} = 0$ we have

$$\dot{\theta} = \eta$$
$$\dot{\eta} = -G'_{\theta}(\theta)|_{\theta=0,\eta=0}\theta + F(\theta,\kappa^{esc})|_{\theta=0,\eta=0}\eta =$$
$$= -\frac{g(M+2m)}{Ml}\theta + \frac{\kappa^{esc}(M+2m)}{Mml}\eta$$

Since

$$\frac{\kappa^{esc}(M+2m)}{Mml} > 0,$$

this implies the instability of zero solution and the unwinding of phase trajectory for small θ.

Following the strategy described in [Pogromsky *et. al.,* 2005; Bennet *et. al.,* 2005], we consider the question of the existence and setting of an in-phase regime.

3.1.3 Proof of the existence of a periodic regime

Having performed some transformation in system (183) and taking into account the relations

$$\kappa = \sqrt{g/l}, \quad \varepsilon_M = \frac{m}{M+2m},$$

$$\delta_M(\theta) = \varepsilon_M(\ddot{\theta}_2\cos^2\theta - \dot{\theta}^2\sin 2\theta),$$

$$f^{esc}_{ml}(\theta) = \kappa^{esc}_{ml}(1-\frac{\theta^2}{\Phi^2}),$$

we obtain

$$\ddot{\theta} + \kappa^2\sin\theta = f^{esc}_{ml}(\theta)\dot{\theta} + \delta_M(\theta) \tag{188}$$

In the approximation when $\sin\theta \approx \theta$, $0 \le \theta < \pi/3$ equation (188) takes the form

$$\ddot{\widetilde{\theta}} + \kappa^2\widetilde{\theta} = f^{esc}_{ml}(\widetilde{\theta})\dot{\widetilde{\theta}} + \delta_M(\widetilde{\theta}). \tag{189}$$

The solution of this equation with the initial states $\widetilde{\theta}(0) = \theta(0) = \theta_0$, $\dot{\widetilde{\theta}}(0) = \dot{\theta}(0) = 0$ can be represented as

$$\widetilde{\theta}(t) = \theta_0\cos(\kappa t) + g_1(t,\kappa^{esc},M),$$

$$\dot{\widetilde{\theta}}(t) = -\theta_0\kappa\sin(\kappa t) + g_2(t,\kappa^{esc},M).$$

For small κ^{esc} and $1/M$ for the time T of second crossing of the trajectory $(\widetilde{\theta}, \dot{\widetilde{\theta}})$ of the straight line $\dot{\theta} = 0$, we approximately obtain

$$T \approx \widetilde{T} = \frac{2\pi}{\kappa} : \ \dot{\widetilde{\theta}}(T) = \dot{\widetilde{\theta}}(0) = 0$$

Let us consider now the Lyapunov function $V(\dot{\widetilde{\theta}}, \widetilde{\theta})$ and its derivative by virtue of system (189), i.e.

$$V(\dot{\widetilde{\theta}}, \widetilde{\theta}) = \dot{\widetilde{\theta}}^2/2 + \frac{\kappa}{2}\widetilde{\theta}^2 > 0$$

and

$$\dot{V}(\dot{\widetilde{\theta}}, \widetilde{\theta}) = \dot{\widetilde{\theta}}(\ddot{\widetilde{\theta}} + \kappa\widetilde{\theta}) = \dot{\widetilde{\theta}}^2 f_{ml}^{esc}(\widetilde{\theta}) + \dot{\widetilde{\theta}}\delta_M(\widetilde{\theta}),$$

respectively. Following [Leonov, 2008], we then estimate the relation

$$V(\dot{\widetilde{\theta}}(T), \widetilde{\theta}(T)) - V(\dot{\widetilde{\theta}}(0), \widetilde{\theta}(0)) = \int_0^T \dot{V}(\dot{\widetilde{\theta}}(t), \widetilde{\theta}(t))\mathrm{d}t \approx \int_0^{\widetilde{T}} \dot{V}(\dot{\widetilde{\theta}}(t), \widetilde{\theta}(t))\mathrm{d}t.$$

For this purpose we integrate $\dot{\widetilde{\theta}}^2 f_{ml}^{esc}\big(\widetilde{\theta}(\tau)\big)$ from 0 to $\widetilde{T}$:

$$\int_0^{\widetilde{T}} \dot{\widetilde{\theta}}^2 \kappa_{ml}^{esc}(1 - \frac{\widetilde{\theta}^2(\tau)}{\Phi^2})\mathrm{d}\tau = \kappa_{ml}^{esc}\theta_0^2\kappa^2 \int_0^{\widetilde{T}} \sin^2(\kappa\tau)\big(1 - \frac{\theta_0^2\cos^2(\kappa\tau)}{\Phi^2}\big)\mathrm{d}\tau + \widetilde{g}_1(\kappa^{esc}, M) =$$
$$= \kappa_{ml}^{esc}\theta_0^2\kappa^2\frac{\pi(4\Phi^2 - \theta_0^2)}{4\kappa\Phi^2} + \widetilde{g}_1(\kappa^{esc}, M)$$

where

$$\lim_{\kappa^{esc}, 1/M \to 0} \widetilde{g}_1(\kappa^{esc}, M) = 0.$$

For the approximate values

$$\cos\widetilde{\theta}(t) \approx \cos\big(\theta_0\cos(\kappa t)\big), \quad \sin\big(2\widetilde{\theta}(t)\big) \approx \sin\big(\theta_0 2\cos(\kappa t)\big),$$

the following relations hold

$$\int_0^{\widetilde{T}} \dot{\widetilde{\theta}}(t)\ddot{\widetilde{\theta}}(t)\cos^2(\widetilde{\theta}(t))\mathrm{d}t = \widetilde{g}_2(\kappa^{esc}, M)$$

$$\int_0^{\widetilde{T}} \dot{\widetilde{\theta}}^3(t)\sin(2\widetilde{\theta}(t))\mathrm{d}t = \widetilde{g}_3(\kappa^{esc}, M),$$

$$\lim_{\kappa^{esc}, 1/M \to 0} \widetilde{g}_i(\kappa^{esc}, M) = 0.$$

Taking into account these relations, we integrate $\dot{\widetilde{\theta}}\delta_M\big(\widetilde{\theta}(\tau)\big)$ from 0 to $\widetilde{T}$

$$\begin{aligned}&\int\limits_0^{\widetilde{T}} \dot{\widetilde{\theta}}(t)\delta_M\big(\widetilde{\theta}(t)\big)\,\mathrm{d}t = \int\limits_0^{\widetilde{T}} \dot{\widetilde{\theta}}(t)\varepsilon_M\big[\ddot{\widetilde{\theta}}(t)2\cos^2\widetilde{\theta}(t) - \dot{\widetilde{\theta}}^2(t)\sin 2\widetilde{\theta}(t)\big]\,\mathrm{d}t = \\ &= \int\limits_0^{\widetilde{T}} -\theta_0\kappa\sin(\kappa t)\varepsilon_M\big[-\theta_0\kappa^2\cos(\kappa t)2\cos\big(\theta_0\cos(\kappa t)\big) - \\ &-\theta_0^2\kappa^2\sin^2(\kappa t)\sin\big(\theta_0 2\cos(\kappa t)\big)\big]\,\mathrm{d}t + \widetilde{g}_4(\kappa^{esc}, M) = 0 + g_4(\kappa^{esc}, M),\end{aligned}$$

where

$$\lim_{\kappa^{esc},1/M\to 0} g(\kappa^{esc}, M) = 0.$$

Then we obtain

$$\begin{aligned}&V(\dot{\widetilde{\theta}}(T), \widetilde{\theta}(T)) - V(\dot{\widetilde{\theta}}(0), \widetilde{\theta}(0)) \approx \\ &\approx \int\limits_0^{\widetilde{T}} \dot{V}(\dot{\theta}(\tau), \theta(\tau))\mathrm{d}\tau = \\ &= \int\limits_0^{\widetilde{T}} \big[\dot{\widetilde{\theta}}^2 f_{ml}^{esc}\big(\widetilde{\theta}(\tau)\big) + \dot{\widetilde{\theta}}\delta_M\big(\widetilde{\theta}(\tau)\big)\big]\,\mathrm{d}\tau = \\ &= \int\limits_0^{\widetilde{T}} \dot{\widetilde{\theta}}^2 f_{ml}^{esc}\big(\widetilde{\theta}(\tau)\big)\,\mathrm{d}\tau + \int\limits_0^{\widetilde{T}} \dot{\widetilde{\theta}}\delta_M\big(\widetilde{\theta}(\tau)\big)\,\mathrm{d}\tau = \\ &= \kappa_{ml}^{esc}\theta_0^2\kappa^2\frac{\pi(4\Phi^2 - \theta_0^2)}{4\kappa\Phi^2} + g(\kappa^{esc}, M),\end{aligned}$$

where

$$\lim_{\kappa^{esc},1/M\to 0} g(\kappa^{esc}, M) = 0.$$

In this case for $-2\Phi < \theta_0 < 2\Phi$ we have an unwinding and for $2\Phi < |\theta_0|$ a twisting of the phase trajectory for sufficiently small $\kappa^{esc}, 1/M$. Thus, it is proved the existence of in-phase regime, i.e. we proved that for $\phi_1 - \phi_2 = 0$, for the sum of the angles $\phi_1 + \phi_2$ there occurs a periodic regime.

3.2 Phase-locked loops

3.2.1 Introduction

The phase-locked loops are widespread in a modern radio electronics and circuit technology [Viterbi, 1966; Gardner, 1966; Lindsey, 1972; Lindsey and Chie, 1981; Leonov *et al.*, 1992; Leonov *et al.*, 1996; Leonov & Smirnova, 2000; Kroupa, 2003; Best, 2003, Razavi, 2003; Egan, 2000; Abramovitch, 2002]. In this study the technique of PLL description on three levels is suggested:

1) on the level of electronic realizations,

2) on the level of phase and frequency relations between inputs and outputs in block diagrams,

3) on the level of differential and integro-differential equations.

The second level, involving the asymptotical analysis of high-frequency oscillations, is necessary for the well-formed derivation of equations and for the passage to the third level of description. For example, the main for the PLL theory notion of phase detector is formed exactly on the second level of consideration. In this case the characteristic of phase detector depends on the class of considered oscillations. While in the classical PLL it is used the oscillation multipliers, for harmonic oscillations the characteristic of phase detector is also harmonic, for the impulse oscillations (for the same electronic realization of feedback loop) that is a continuous piecewise-linear periodic function.

In the present work the development of the above-mentioned technique of PLL is proposed. Here for the standard electronic realizations, the characteristics of phase detectors are computed and the differential equations, describing the PLL operation, are derived.

Here the classical ideas by Viterbi [Viterbi, 1966] are extended and generalized for design of PLL with pulse modulation. Introduction of a relay element in the block diagram after filter is essentially new construction for floating PLL with respect to previous design of floating PLL for radio engineering [Viterbi, 1966]. In this work it is shown that the main requirement to PLL for multiprocessor systems is global stability. Necessary and sufficient conditions of global stability for floating PLL are obtained. For the proof of these results the direct Lyapunov method is applied.

3.2.2 Block diagram and mathematical model of PLL

Consider a PLL on the first level (Fig. 8)

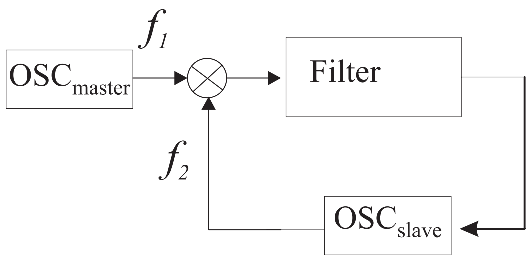

FIGURE 8 Electronic circuit of PLL

Here OSC_{master} is a master oscillator, OSC_{slave} is a slave oscillator, which generates high-frequency "almost harmonic oscillations"

$$f_j(t) = A_j \sin(\omega_j(t)t + \psi_j). \tag{190}$$

Block $\otimes$ is a multiplier of oscillations of $f_1(t)$ and $f_2(t)$. At its output the signal $f_1(t)f_2(t)$ occurs. The relations between the input $\xi(t)$ and the output $\sigma(t)$

of linear filter have the form

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\, d\tau.$$

Here $\gamma(t)$ is an impulse transient function of filter, $\alpha_0(t)$ is an exponentially damped function, depending on the initial data of filter at the moment $t = 0$.

Now we reformulate the high-frequency property of oscillations $f_j(t)$ to obtain the following condition.

Consider the great fixed time interval $[0, T]$, which can be partitioned into small intervals of the form $[\tau, \tau + \delta]$, $(\tau \in [0, T])$, where the following relations

$$\begin{gathered} |\gamma(t) - \gamma(\tau)| \le C\delta, \ \ |\omega_j(t) - \omega_j(\tau| \le C\delta, \\ \forall\, t \in [\tau, \tau + \delta], \ \ \forall\, \tau \in [0, T], \end{gathered} \tag{191}$$

$$|\omega_1(\tau) - \omega_2(\tau)| \le C_1, \ \ \forall\, \tau \in [0, T], \tag{192}$$

$$\omega_j(t) \ge R, \ \ \forall\, t \in [0, T] \tag{193}$$

are satisfied. Here we assume that the quantity δ is sufficiently small with respect to the fixed numbers T, C, C_1, the number R is sufficiently great with respect to the number δ.

The latter means that on the small intervals $[\tau, \tau + \delta]$ the functions $\gamma(t)$ and $\omega_j(t)$ are "almost constants" and the functions $f_j(t)$ rapidly oscillate as harmonic functions. It is clear that such conditions occur for high-frequency oscillations.

Consider two block diagrams described in Fig. 9 and 10.

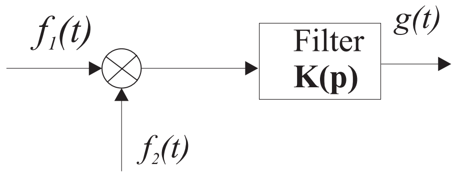

FIGURE 9 Multiplier and filter with transfer function $K(p)$

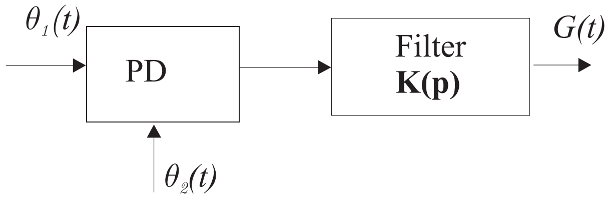

FIGURE 10 Phase detector and filter

Here $\theta_j(t) = \omega_j(t)t + \Psi_j$ are phases of the oscillations $f_j(t)$, PD is a nonlinear block with the characteristic $\varphi(\theta)$ (being called a phase detector or discriminator). The phases $\theta_j(t)$ enter the inputs of PD block and the output is the function $\varphi(\theta_1(t) - \theta_2(t))$.

The signals $f_1(t)f_2(t)$ and $\varphi(\theta_1(t) - \theta_2(t))$ enter the same filters with the same impulse transient function $\gamma(t)$. The filter outputs are the functions $g(t)$ and $G(t)$ respectively.

A classical PLL synthesis is based on the following result [Viterbi, 1966]:

Theorem 14 *If conditions (191)–(193) are satisfied and*

$$\varphi(\theta) = \frac{1}{2}A_1A_2\cos\theta,$$

then for the same initial data of filter the following relation

$$|G(t) - g(t)| \le C_2\delta, \ \forall\, t \in [0, T].$$

is valid. Here C_2 is a certain number not depending on δ.

Thus, the outputs of two block diagrams in Fig. 9 and Fig. 10: $g(t)$ and $G(t)$, respectively, differ little from each other and we can pass (from a standpoint of the asymptotic with respect to δ) to the following description level, namely to the level of phase relations 2).

In this case a block diagram in Fig. 8 passes to the following block diagram (Fig. 11)

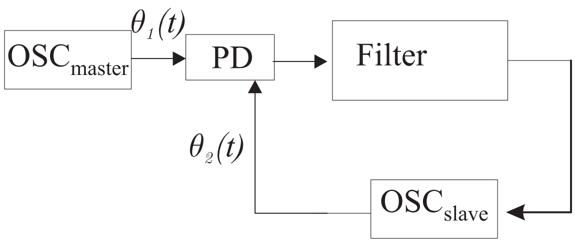

FIGURE 11 Block diagram of PLL on the level of phase relations

Consider now the high-frequency oscillators, connected by a diagram in Fig. 8. Here

$$f_j(t) = A_j\mathrm{sign}\,(\sin(\omega_j(t)t + \psi_j)). \tag{194}$$

We assume, as before, that conditions (191)–(193) are satisfied.

Consider a 2π-periodic function $\varphi(\theta)$ of the form

$$\varphi(\theta) = \begin{cases} A_1A_2(1 + 2\theta/\pi) & \text{for } \theta \in [-\pi, 0], \\ A_1A_2(1 - 2\theta/\pi) & \text{for } \theta \in [0, \pi]. \end{cases} \tag{195}$$

and block diagrams in Fig. 9 and 10.

Theorem 15 *If conditions (191)–(193) are satisfied and the characteristic of phase detector $\varphi(\theta)$ has the form (195), then for the same initial data of filter the following relation holds*

$$|G(t) - g(t)| \leq C_3\delta, \ \forall\, t \in [0, T].$$

Here C_3 is a certain number not depending on δ.

Theorem 15 is a base for the synthesis of PLL with impulse oscillators. It permits us for the impulse clock oscillators to consider two block diagrams in parallel: on the level of electronic realization (Fig. 8) and on the level of phase relations (Fig. 11), where the common principles of the phase synchronization theory can be applied. Thus, we can construct the theory of phase synchronization for the distributed system of clocks in multiprocessor cluster.

Let us make a remark necessary to derive the differential equations of PLL.

Consider a quantity

$$\dot{\theta}_j(t) = \omega_j(t) + \dot{\omega}_j(t)t.$$

For the well-synthesized PLL, namely possessing the property of global stability, we have an exponential damping of the quantity $\dot{\omega}_j(t)$:

$$|\dot{\omega}_j(t)| \leq Ce^{-\alpha t}.$$

Here C and α are certain positive numbers not depending on t. Therefore the quantity $\dot{\omega}_j(t)t$ is, as a rule, sufficiently small with respect to the number R (see condition (191)– (193)).

From the above we can conclude that the following approximate relation

$$\dot{\theta}_j(t) = \omega_j(t) \tag{196}$$

is valid. When derived the differential equations of this PLL, we make use of a block diagram in Fig. 11 and relation (196), which is assumed to be valid precisely.

Note that, by assumption, the control law of tunable oscillators is linear:

$$\omega_2(t) = \omega_2(0) + LG(t). \tag{197}$$

Here $\omega_2(0)$ is the initial frequency of tunable oscillator, L is a certain number, and $G(t)$ is a control signal, which is a filter output (Fig. 11).

Thus, the equation of PLL is as follows

$$\dot{\theta}_2(t) = \omega_2(0) + L(\alpha_0(t) + \int\limits_0^t \gamma(t-\tau).\ \varphi(\theta_1(\tau) - \theta_2(\tau))d\tau).$$

Assuming that the master oscillator is such that $\omega_1(t) \equiv \omega_1(0)$, we obtain the following relations for PLL

$$\begin{gathered}(\theta_1(t) - \theta_2(t))^{\bullet} + L(\alpha_0(t) + \int\limits_0^t \gamma(t-\tau). \\ \varphi(\theta_1(\tau) - \theta_2(\tau))d\tau) = \omega_1(0) - \omega_2(0).\end{gathered} \tag{198}$$

This is an equation of PLL.

By a similar approach we can conclude that in PLL it can be used the filters with transfer functions of more general form

$$K(p) = a + W(p),$$

where a is a certain number, $W(p)$ is a proper fractional rational function. In this case in place of equation (198) we have

$$\begin{aligned}
&(\theta_1(t) - \theta_2(t))^\bullet + L(a(\varphi(\theta_1(t) - \theta_2(t)) + \\
&+ \alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi(\theta_1(\tau) - \theta_2(\tau))d\tau) = \\
&= \omega_1(0) - \omega_2(0).
\end{aligned} \tag{199}$$

In the case when the transfer function of the filter $a + W(p)$ is non-degenerate, i.e. its numerator and denominator do not have common roots, equation (199) is equivalent to the following system of differential equations

$$\begin{aligned}
\dot{z} &= Az + b\psi(\sigma) \\
\dot{\sigma} &= c^* z + \rho\psi(\sigma).
\end{aligned} \tag{200}$$

Here A is a constant $(n \times n)$-matrix, b and c are constant $(n \times n)$-vectors, ρ is a number, and $\psi(\sigma)$ is a 2π-periodic function, satisfying the relations:

$$\begin{aligned}
&\rho = -aL, \\
&W(p) = L^{-1}c^*(A - pI)^{-1}b, \\
&\psi(\sigma) = \varphi(\sigma) - \frac{\omega_1(0) - \omega_2(0)}{L(a + W(0))}.
\end{aligned}$$

Note that in (11) $\sigma = \theta_1 - \theta_2$.

Using Theorem 15, we can make the design of a block diagram of floating PLL, which plays a role of the function of frequency synthesizer and the function of correction of the clock-skew (see parameter τ in Fig. 12).

Such a block diagram is shown in Fig. 12.

Here $\text{OSC}_{\text{master}}$ is a master oscillator, *Delay* is a time-delay element, *Filter* is a filter with transfer function

$$W(p) = \frac{\beta}{p + \alpha},$$

$\text{OSC}_{\text{slave}}$ is a slave oscillator, PD1 and PD2 are programmable dividers of frequencies, and *Processor* is a processor.

The *Relay* element plays a role of a floating correcting block. The introduction of it allows us to null a residual clock skew, which arises for the nonzero initial difference of frequencies of master and slave oscillators.

Note that the electronic realization of clock and delay can be found in [Ugrumov, 2000; Razavi, 2003] and that of multipliers, filters, and relays in [Aleksenko,

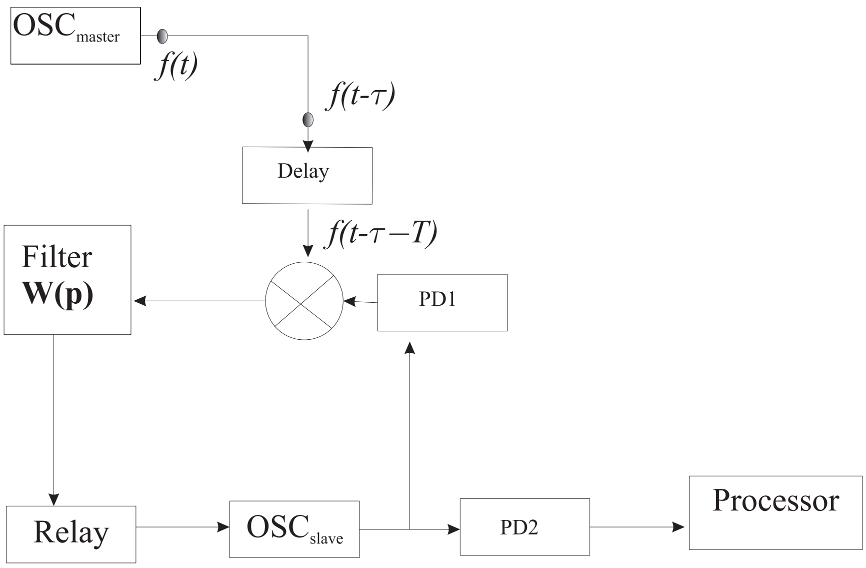

FIGURE 12 Block diagram of PLL

2004; Razavi, 2003]. The description of dividers of frequency can be found in [Solonina *et al.*, 2000].

Assume, as usual, that the frequency of master oscillator is constant, namely $\omega_1(t) \equiv \omega_1 = const$. The parameter of delay line T is chosen in such a way that $\omega_1(T+\tau) = 2\pi k + 3\pi/2$. Here k is a certain natural number, $\omega_1\tau$ is a clock skew.

By Theorem 15 and the choice of T the block diagram, shown in Fig. 12, can be changed by the close block diagram, shown in Fig. 13.

Here $\varphi(\theta)$ is a 2π-periodic characteristic of phase detector. It has the form

$$\varphi(\theta) = \begin{cases} 2A_1A_2\theta/\pi & \text{for } \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 2A_1A_2(1-\theta/\pi) & \text{for } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}], \end{cases} \tag{201}$$

$\theta_2(t) = \dfrac{\theta_3(t)}{M}, \theta_4(t) = \dfrac{\theta_3(t)}{N}$, where the natural numbers M and N are parameters of programmable divisions PD1 and PD2, respectively.

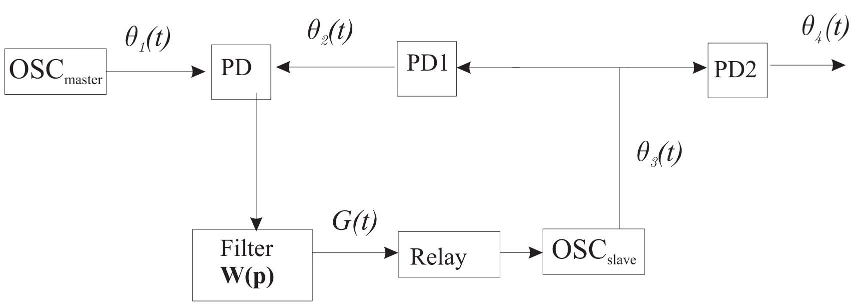

FIGURE 13 Equivalent block diagram of PLL

For a transient process (a capture mode) the following conditions:

$$\lim_{t\to+\infty}(\theta_4(t) - \frac{M}{N}\theta_1(t)) = \frac{2\pi kM}{N} \qquad (202)$$

(a phase capture) and

$$\lim_{t\to+\infty}(\dot\theta_4(t) - \frac{M}{N}\dot\theta_1(t)) = 0 \qquad (203)$$

(a frequency capture), must be satisfied.

Relations (202) and (203) are the main requirements to PLL for array processors. The time of transient processors depends on the initial data and is sufficiently large for multiprocessor system [Leonov & Seledzhi, 2002; Kung, 1988].

Assuming that the characteristic of relay is of the form $\Psi(G) = \mathrm{sign}G$ and the actuating element of slave oscillator is linear, we have

$$\dot\theta_3(t) = R\mathrm{sign}G(t) + \omega_3(0), \qquad (204)$$

where R is a certain number, $\omega_3(0)$ is the initial frequency, and $\theta_3(t)$ is a phase of slave oscillator.

Taking into account relations (204), (190), (201) and the block diagram in Fig. 13, we have the following differential equations of PLL

$$\begin{aligned}&\dot G + \alpha G = \beta\varphi(\theta)\\ &\dot\theta = -\frac{R}{M}\mathrm{sign}G + (\omega_1 - \frac{\omega_3(0)}{M}).\end{aligned} \qquad (205)$$

Here $\theta(t) = \theta_1(t) - \theta_2(t)$.

3.2.3 Criterion of global stability of PLL

Rewrite system (205) as follows

$$\begin{aligned}&\dot G = -\alpha G + \beta\varphi(\theta)\\ &\dot\theta = -F(G),\end{aligned} \qquad (206)$$

where

$$F(G) = \frac{R}{M}\mathrm{sign}G - (\omega_1 - \frac{\omega_3(0)}{M}).$$

Theorem 16 *If the inequality*

$$|R| > |M\omega_1 - \omega_3(0)| \qquad (207)$$

is valid, then any solution of system (206) tends to a certain equilibrium as $t \to +\infty$.

If the inequality

$$|R| < |M\omega_1 - \omega_3(0)| \qquad (208)$$

is valid, then all the solutions of system (206) tends to infinity as $t \to +\infty$.

Consider equilibria for system (206). For any equilibrium we have

$$\dot{\theta}(t) \equiv 0, \quad G(t) \equiv 0, \quad \theta(t) \equiv \pi k.$$

Theorem 17 *Assume that relation (207) is valid. In this case if $R > 0$, then the following equilibria*

$$G(t) \equiv 0, \quad \theta(t) \equiv 2k\pi \tag{209}$$

are locally asymptotically stable and the following equilibria

$$G(t) \equiv 0, \quad \theta(t) \equiv (2k+1)\pi \tag{210}$$

are locally unstable. If $R < 0$, then equilibria (210) are locally asymptotically stable and equilibria (209) are locally unstable.

Thus, for relations (202) and (203) to be satisfied it is necessary to choose the parameters of system in such a way that the inequality holds

$$R > |M\omega_1 - \omega_3(0)|. \tag{211}$$

3.2.4 Proofs of theorems

Proof of Theorem 14

For $t \in [0, T]$, we obviously have

$$\begin{aligned} g(t) - G(t) = \int\limits_0^t \gamma(t-s)\,[A_1 A_2 \sin(\omega_1(s)s - \psi_1)\cdot \\ \sin(\omega_2(s)s + \psi_2) - \varphi(\omega_1(s)s - \omega_2(s)s + \psi_1 - \psi_2]\,ds = \\ = -\frac{A_1 A_2}{2}\int\limits_0^t \gamma(t-s)\,[\cos\left((\omega_1(s) + \omega_2(s))s + \psi_1 + \psi_2\right)]\,ds. \end{aligned}$$

Consider the intervals $[k\delta, (k+1)\delta]$, where $k = 0, \ldots, m$ and the number m is such that

$$t \in [m\delta, (m+1)\delta].$$

From conditions (191)–(193) it follows that for any $s \in [k\delta, (k+1)\delta]$ the relations

$$\gamma(t-s) = \gamma(t-k\delta) + O(\delta) \tag{212}$$

$$\omega_1(s) + \omega_2(s) = \omega_1(k\delta) + \omega_2(k\delta) + O(\delta) \tag{213}$$

are valid on each interval $[k\delta, (k+1)\delta]$. Then by (213) for any $s \in [k\delta, (k+1)\delta]$ the estimate

$$\begin{aligned} \cos\left((\omega_1(s) + \omega_2(s))s + \psi_1 + \psi_2\right) = \\ = \cos\left((\omega_1(k\delta) + \omega_2(k\delta))s + \psi_1 + \psi_2\right) + O(\delta) \end{aligned} \tag{214}$$

is satisfied.

Relations (212) and (214) imply that

$$\int_0^t \gamma(t-s)\left[\cos\left((\omega_1(s)+\omega_2(s))s+\psi_1+\psi_2\right)\right] ds =$$
$$= \sum_{k=0}^{m} \gamma(t-k\delta) \int_{k\delta}^{(k+1)\delta} \left[\cos\left((\omega_1(k\delta)+\omega_2(k\delta))s+\psi_1+\psi_2\right)\right] ds + O(\delta). \quad (215)$$

From (193) we have the estimate

$$\int_{k\delta}^{(k+1)\delta} \left[\cos\left((\omega_1(k\delta)+\omega_2(k\delta))s+\psi_1+\psi_2\right)\right] ds = O(\delta^2)$$

and the fact that R is sufficiently great (Fig. 14) as compared with δ. Then we

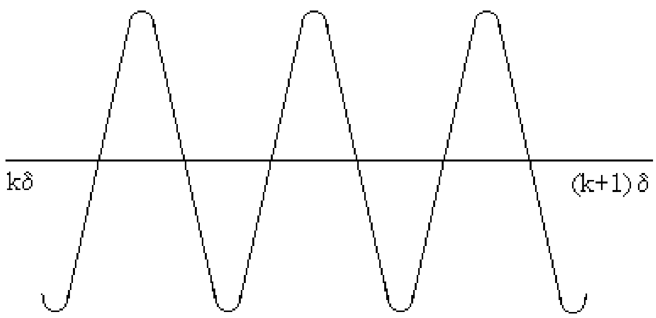

FIGURE 14 High frequency oscillation

have

$$\int_0^t \gamma(t-s)\left[\cos\left((\omega_1(s)+\omega_2(s))s+\psi_1+\psi_2\right)\right] ds = O(\delta).$$

■

Proof of Theorem 15

It is well known that for a filter with the impulse transition function $\gamma(t)$, input $\varepsilon(t)$, output $\sigma(t)$, and eigen oscillation $\alpha(t)$, the following relation holds

$$\sigma(t) = \alpha(t) + \int_0^t \gamma(t-s)\xi(s)\, ds.$$

In this case the formula holds

$$g(t) - G(t) = \int_0^t \gamma(t-s)[A_1 A_2 \mathrm{sign}[\sin(\omega_1(s)s+$$
$$+\psi_1)\sin(\omega_2(s)s+\psi_2)]-$$
$$-\varphi(\omega_1(s)s - \omega_2(s)s + \psi_1 - \psi_2)]\,ds.$$

Partitioning the interval $[0,t]$ into the intervals $[k\delta,(k+1)\delta]$ and making use of assumptions (193) and (194), we replace the above integral with the following sum

$$\sum_{k=0}^{m} \gamma(t-k\delta)[\int_{k\delta}^{(k+1)\delta} A_1 A_2 \mathrm{sign}[\cos((\omega_1(k\delta)-$$
$$\omega_2(k\delta))k\delta + \psi_1 - \psi_2) - \cos((\omega_1(k\delta)+$$
$$+\omega_2(k\delta))s + \psi_1 + \psi_2)]\,ds - \varphi((\omega_1(k\delta)-$$
$$-\omega_2(k\delta))k\delta + \psi_1 - \psi_2)\delta].$$

The number m is chosen in such a way that $t \in [m\delta,(m+1)\delta]$. Since $(\omega_1(k\delta) + \omega_2(k\delta))\delta \gg 1$, the relation

$$\int_{k\delta}^{(k+1)\delta} A_1 A_2 \mathrm{sign}[\cos((\omega_1(k\delta) - \omega_2(k\delta))k\delta + \psi_1 - \psi_2)-$$
$$-\cos((\omega_1(k\delta)+\omega_2(k\delta))s + \psi_1 + \psi_2)]\,ds \approx$$
$$\approx \varphi((\omega_1(k\delta) - \omega_2(k\delta))k\delta + \psi_1 - \psi_2)\delta, \qquad (216)$$

is satisfied. Here we applied the relation

$$A_1 A_2 \int_{k\delta}^{(k+1)\delta} \mathrm{sign}[\cos\alpha - \cos(\omega s + \psi_0)]\,ds \approx \varphi(\alpha)\delta$$

for $\omega\delta \gg 1$, $\alpha \in [-\pi,\pi]$, $\psi_0 \in R^1$.

Formula (216) yields inequality (195).

■

To prove Theorem 16, we formulate an extension of Barbashin–Krasovsky theorem to dynamical systems with a cylindrical phase space. Consider the differential inclusion

$$\frac{dx}{dt} \in f(x), \quad x \in R^n, \quad t \in R^1, \qquad (217)$$

where $f(x)$ is a semicontinuous vector function whose values are the bounded closed convex sets $f(x) \subset R^n$. Here R^n is an n-dimensional Euclidean space. Recall now the basic definitions of the theory of differential inclusions.

Definition 14 *We say that $U_\varepsilon(\Omega)$ is an ε-neighborhood of the set Ω if*

$$U_\varepsilon(\Omega) = \{x \mid \inf_{y\in\Omega} |x-y| < \varepsilon\},$$

where $|\cdot|$ is the Euclidean norm in R^n.

Definition 15 *A function $f(x)$ is called semicontinuous at a point x if for any $\varepsilon > 0$ there exists a number $\delta(x, \varepsilon) > 0$ such that the following containment holds*

$$f(y) \in U_\varepsilon(f(x)), \ \ \forall\, y \in U_\delta(x).$$

Definition 16 *A vector function $x(t)$ is called a solution of differential inclusion if it is absolutely continuous and for the values of t such that the derivative $\dot{x}(t)$ exists, the inclusion*

$$\dot{x}(t) \in f(x(t))$$

is valid.

Under the above assumptions on the function $f(x)$ the theorem on the existence and continuability of solution of differential inclusion (217) holds [Yakubovich *et al.*, 2004]. We assume that linearly independent vectors $d_1, \ldots, d_m$ satisfy the relation

$$f(x + d_j) = f(x), \ \ \forall\, x \in R^n. \tag{218}$$

As a rule, $d_j^* x$ is called a phase or angular coordinate of system (217). Since property (218) allows us to introduce a cylindrical phase space [Yakubovich *et al.*, 2004], system (217) with property (218) is often called a system with cylindrical phase space.

The following theorem is an extension of the well–known Barbashin–Krasovsky theorem to differential inclusions with cylindrical phase space.

Theorem 18 *Suppose that there exists a continuous function $V(x) : R^n \to R^1$ such that the following conditions hold:*

1) $V(x + d_j) = V(x), \ \ \forall x \in R^n, \ \ \forall j = 1, \ldots, m$;

2) $V(x) + \sum\limits_{j=1}^{m} (d_j^* x)^2 \to \infty$ as $|x| \to \infty$;

3) for any solution $x(t)$ of inclusion (217) the function $V(x(t))$ is nonincreasing;

4) if $V(x(t)) \equiv V(x(0))$, then $x(t)$ is an equilibrium.

Then, any solution of inclusion (217) tends to a stationary set as $t \to +\infty$.

Recall that if the solution tends to the stationary set Λ as t, we have

$$\lim_{t \to +\infty} \inf_{z \in \Lambda} |z - x(t)| = 0.$$

A proof of Theorem 18 can be found in [Yakubovich *et al.*, 2004].

Proofs of Theorems 16 and 17

Let $R > |M\omega_1 - \omega_2(0)|$. Consider the Lyapunov function

$$V(G, \theta) = \int_0^G \Phi(u)du + \beta \int_0^\theta \varphi(u)du,$$

where $\Phi(G)$ is a single-valued function coinciding with $F(G)$ for $G \neq 0$. At the point $G = 0$ the function $\Phi(G)$ can be defined arbitrary. At the points t such that $G(t) \neq 0$, we have

$$\frac{dV(G(t), \theta(t))}{dt} = -\alpha G(t) F(G(t)). \tag{219}$$

Note that for $G(t) = 0$ the first equation of system (201) becomes

$$\dot{G}(t) \neq 0 \text{ for } \theta(t) \neq k\pi.$$

It follows that there are no sliding solutions of system (201). Then relation (219) and the inequality $F(G)G > 0, \ \forall G \neq 0$ result in the fact that conditions (192) and (193) of Theorem 18 are satisfied. Moreover $V(G, \theta + 2\pi) \equiv V(G, \theta)$ and $V(G, \theta) \to +\infty$ as $G \to +\infty$. Then conditions (190) and (191) of Theorem 5 are satisfied. Hence any solution of system (201) tends to a stationary set as $t \to +\infty$. Since a stationary set of system (201) consists of isolated points, any solution of system (201) tends to equilibrium as $t \to +\infty$.

If the inequality

$$-R > |M\omega_1 - \omega_3(0)| \tag{220}$$

is valid, then in place of the function $V(G, \theta)$ we can consider the Lyapunov function $W(G, \theta) = -V(G, \theta)$ and repeat the above considerations.

Due to inequality (220) we have the relation $F(G) \neq 0, \forall G \in R^1$. Together with the second equation of system (201) this implies that

$$\lim_{t \to +\infty} \theta(t) = \infty.$$

Thus, Theorem 16 is completely proved.

To prove Theorem 17, we note that if condition (207) is valid in a neighborhood of the points $G = 0, \theta = 2\pi k$, then the function $V(G, \theta)$ has the property

$$V(G, \theta) > 0 \quad \text{for} \quad |G| + |\theta - 2k\pi| \neq 0.$$

Taking into account (219), from the above we obtain the asymptotic stability of these equilibria.

In a neighborhood of the points $G = 0, \theta = (2k+1)\pi k$ the function $V(G, \theta)$ has the property

$$V(0, \theta) < 0 \quad \text{for} \quad \theta \neq (2k+1)\pi.$$

Then by (219) we obtain the instability of these equilibria.

If inequality (220) holds, then in place of the function $V(G, \theta)$ we make use of the function $W(G, \theta) = -V(G, \theta)$ and repeat the considerations.

■

YHTEENVETO (FINNISH SUMMARY)

Käsillä oleva työ keskittyy diskreettien ja jatkuvien dynaamisten systeemien kvalitatiivisen teorian kysymyksiin ja niiden sovelluksiin.

Ensimmäisessä luvussa tarkastellaan approksimatiivista menetelmää diskreettien ja jatkuvien dynaamisten systeemien stabiiliuden ja epästabiiliuden tutkimiseksi seuraten A.M. Lyapunovin, O. Perronin ja N.G. Chetaevin töitä. Ajassa muuttuvan liikkeen ensimmäisen kertaluvun approksimaation klassinen stabiilisuusongelma todistetaan yleisessä muodossa. Kappaleessa tutkitaan myös alkuperäisen systeemin ja ensimmäisen kertaluvun approksimoivan systeemin ratkaisujen karakteristisen eksponentin merkkimuutoksen Perronin efektejä samoilla lähtöarvoilla.

Toisessa luvussa tarkastellaan kaksiulotteisen autonomisen systeemin kvalitatiivista käyttäytymistä tilanteessa, jossa ensimmäisen kertaluvun approksimaation systeemillä on kaksi puhtaasti imaginääristä ominaisarvoa. Tässä käytetään klassista Poincaren-Lyapunovin menetelmää Lyapunov-lukujen laskemiseksi, jotka määrittelevät lentoratojen (kiertyvä tai suoristuva) kvalitatiivisen käyttäytymisen tasossa. Uusi menetelmä, jossa ei tarvita muunnosta normaalimuotoon, kehitetään Lyapunov-lukujen laskemiseksi Euklidisissa koordinaateissa ja aika-alueella ja sille löydetään sovelluskohteita. Tämän menetelmän edut syntyvät sen ideologisesta yksinkertaisuudesta ja visuaalisesta voimasta. Yleiset kaavat kolmannen Lyapunov-luvun esittämiseksi alkuperäisen systeemin kertoimien avulla on löydetty käyttäen apuna moderneja symbolisen laskennan tietokoneohjelmia.

Kolmas luku liittyy faasisynkronisaation matemaattisten mallien differentiaaliyhtälöiden kvalitatiivisen teorian sovelluksiin, kuten yhdistettyjen heilurien systeemeihin Huygensin ongelmassa ja systeemien faasilukittujen silmukoiden taajuuskontrollointiin.

APPENDIX 1 COMPUTATION OF THE FIRST, SECOND, AND THIRD LYAPUNOV QUANTITIES IN THE GENERAL FORM

Consider a complete system with the expansion of the right-hand side up to the seventh order

$$
\begin{aligned}
\dot{x} = \; & -y + f_{20}x^2 + f_{11}xy + f_{02}y^2 + f_{30}x^3 + f_{21}x^2y + f_{12}xy^2 + f_{03}y^3 + \\
& + f_{40}x^4 + f_{31}x^3y + f_{22}x^2y^2 + f_{13}xy^3 + f_{04}y^4 + \\
& + f_{50}x^5 + f_{41}x^4y + f_{32}x^3y^2 + f_{23}x^2y^3 + f_{14}xy^4 + f_{05}y^5 + \\
& + f_{60}x^6 + f_{51}x^5y + f_{42}x^4y^2 + f_{33}x^3y^3 + f_{24}x^2y^4 + f_{15}xy^5 + f_{06}y^6 \\
& + f_{70}x^7 + f_{61}x^6y + f_{52}x^5y^2 + f_{43}x^4y^3 + f_{34}x^3y^4 + f_{25}x^2y^5 + f_{16}xy^6 + f_{07}y^7 + \\
& + o((|x| + |y|)^7), \\
\dot{y} = \; & x + g_{20}x^2 + g_{11}xy + g_{02}y^2 + g_{30}x^3 + g_{21}x^2y + g_{12}xy^2 + g_{03}y^3 + \\
& + g_{40}x^4 + g_{31}x^3y + g_{22}x^2y^2 + g_{13}xy^3 + g_{04}y^4 + \\
& + g_{50}x^5 + g_{41}x^4y + g_{32}x^3y^2 + g_{23}x^2y^3 + g_{14}xy^4 + g_{05}y^5 + \\
& + g_{60}x^6 + g_{51}x^5y + g_{42}x^4y^2 + g_{33}x^3y^3 + g_{24}x^2y^4 + g_{15}xy^5 + g_{06}y^6 \\
& + g_{70}x^7 + g_{61}x^6y + g_{52}x^5y^2 + g_{43}x^4y^3 + g_{34}x^3y^4 + g_{25}x^2y^5 + g_{16}xy^6 + g_{07}y^7 + \\
& + o((|x| + |y|)^7).
\end{aligned}
\tag{221}
$$

For the first Lyapunov quantity we have [Bautin, 1949,1952]

$$L_1 = \frac{\pi}{4}(g_{21} + f_{12} + 3f_{30} + 3g_{03} + f_{20}f_{11} + f_{02}f_{11} - g_{11}g_{20} + 2g_{02}f_{02} - 2f_{20}g_{20} - g_{02}g_{11}).$$

Note that since $\widetilde{T}_1 = 0$, the residual of crossing time does not influence L_1.

To compute the second Lyapunov quantity, we obtain the coefficients $\widetilde{T}_2$ and $\widetilde{T}_3$ of the expansion of crossing time residual. They are the following

$$
\begin{aligned}
\widetilde{T}_2 = \; & \frac{\pi}{12}(-9g_{30} + 4f_{20}^2 + 9f_{03} - 3g_{12} + 10g_{20}^2 + 10f_{02}^2 + 4g_{02}^2 + g_{11}^2 + f_{11}^2 + 3f_{21} - \\
& -5f_{20}g_{11} - f_{11}g_{20} - 5f_{11}g_{02} + 10g_{02}g_{20} - f_{02}g_{11} + 10f_{20}f_{02}),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{T}_3 = \; & -\frac{\pi}{18}(2f_{20} + f_{02} + g_{11})(-9g_{30} + 4f_{20}^2 + 9f_{03} - 3g_{12} + 10g_{20}^2 + 10f_{02}^2 + 4g_{02}^2 \\
& + g_{11}^2 + f_{11}^2 + 3f_{21} - 5f_{20}g_{11} - f_{11}g_{20} - 5f_{11}g_{02} + 10g_{02}g_{20} - f_{02}g_{11} + 10f_{20}f_{02}).
\end{aligned}
$$

From the condition $L_1 = 0$ we obtain the coefficient g_{03}

$$g_{03} = -\frac{1}{3}(g_{21} + f_{12} + 3f_{30} + f_{20}f_{11} + f_{02}f_{11} - g_{11}g_{20} + 2g_{02}f_{02} - 2f_{20}g_{20} - g_{02}g_{11})$$

and the expression for the second Lyapunov quantity

$L_2 = -\frac{\pi}{72}(-66f_{20}g_{04} - 3f_{11}g_{30}f_{20} - 24g_{20}g_{02}g_{21} + 12f_{30}g_{11}f_{02} + 4f_{11}f_{20}^2g_{11} -$
$12f_{11}f_{21}f_{20} + 2g_{20}g_{11}^3 - 9g_{11}g_{02}g_{12} - 12f_{20}f_{11}f_{03} - 12g_{11}g_{02}f_{03} + 3g_{20}f_{12}f_{11} + 9g_{21}g_{30} -$
$6f_{02}f_{11}g_{12} + 9g_{20}g_{11}g_{02}^2 + 30f_{20}g_{02}g_{12} + 30g_{02}f_{21}f_{20} - 60g_{04}f_{02} + g_{11}^2f_{11}f_{20} - 5f_{11}f_{20}^3 -$
$21f_{20}f_{13} - 3f_{11}^3f_{20} - 9g_{02}g_{21}f_{11} + 7g_{11}g_{21}f_{02} - 5f_{11}g_{11}f_{02}^2 + 5f_{02}^2f_{11}f_{20} - 3g_{11}g_{20}f_{21} +$
$6g_{02}f_{20}f_{11}^2 + 9g_{21}f_{03} - 3f_{30}f_{11}^2 + 15f_{11}f_{40} - 21g_{11}g_{30}g_{02} - 6g_{11}f_{03}f_{11} + f_{11}f_{02}g_{11}^2 -$
$18g_{20}f_{03}f_{20} - 42g_{20}g_{02}f_{30} - 6g_{11}g_{12}g_{20} - 30f_{02}^2g_{20}f_{20} + 3f_{11}^2g_{02}f_{02} + 60f_{40}g_{20} + 9g_{11}g_{40} +$
$24f_{20}g_{20}f_{21} - 9g_{11}g_{20}f_{03} - 10g_{11}f_{20}^2g_{02} + 18g_{02}g_{12}f_{02} - 6g_{11}f_{11}g_{30} - 24f_{20}f_{03}g_{02} -$
$30f_{03}f_{02}g_{02} - 24g_{11}g_{20}g_{30} - 12f_{11}f_{30}g_{02} - 3g_{12}f_{11}f_{20} + f_{12}g_{11}^2 - 9f_{21}f_{30} + 27f_{30}g_{30} +$

$$3f_{30}g_{11}^2+15f_{30}g_{02}^2-9f_{02}f_{31}-28g_{20}f_{02}f_{20}^2-2g_{11}^2g_{21}-3f_{22}f_{11}-14f_{12}f_{20}^2-6f_{12}g_{12}+$$
$$27g_{13}g_{02}-3f_{02}f_{11}^3+7f_{20}g_{21}g_{11}+3g_{20}^2f_{11}f_{20}-10g_{02}g_{11}^2f_{20}-10f_{02}f_{12}f_{20}-12g_{20}f_{30}f_{11}+$$
$$6f_{02}f_{21}g_{20}+18f_{02}f_{11}g_{02}^2+3f_{12}g_{02}f_{11}+6g_{20}g_{02}f_{12}+18g_{02}^2f_{12}+9g_{13}g_{20}-3f_{12}f_{11}^2-$$
$$45g_{20}^2f_{30}-15f_{13}f_{02}+30f_{20}g_{20}^3-18g_{02}f_{04}+18f_{20}g_{40}-21g_{20}f_{20}^2g_{11}+2g_{02}g_{11}^3+3f_{02}g_{20}f_{11}^2+$$
$$20f_{02}f_{20}^2g_{02}-9g_{02}^2g_{21}-9g_{21}g_{20}f_{11}-9f_{04}f_{11}+6f_{22}g_{20}+45f_{30}f_{02}^2+15g_{11}g_{20}^3-15g_{11}g_{04}+$$
$$12f_{02}g_{02}f_{21}-5f_{12}g_{11}f_{20}+18g_{12}g_{20}f_{20}-5f_{12}g_{11}f_{02}+20f_{02}g_{21}f_{20}+21g_{02}g_{31}-30g_{20}g_{02}^2f_{20}+$$
$$6g_{20}g_{12}f_{02}+12f_{22}g_{02}+3f_{21}g_{21}+18f_{20}^3g_{02}+24g_{11}g_{20}^2g_{02}+18f_{20}g_{02}g_{20}^2+6f_{11}g_{31}-$$
$$6g_{22}f_{02}+15g_{31}g_{20}+3g_{22}g_{11}-12g_{22}f_{20}-9f_{30}g_{12}-18f_{20}g_{02}^3-24f_{20}g_{02}g_{30}+15f_{20}f_{11}g_{02}^2-$$
$$7g_{20}f_{02}g_{11}^2+6g_{20}f_{02}f_{11}g_{02}-6g_{11}f_{13}-28f_{02}g_{11}g_{20}f_{20}-12g_{11}f_{20}g_{02}f_{02}+9g_{02}g_{11}f_{11}g_{20}-$$
$$9f_{02}f_{21}f_{11}-f_{11}g_{11}f_{02}f_{20}-15f_{30}f_{20}^2+10f_{02}^2f_{20}g_{02}-8g_{11}^2f_{02}g_{02}+42f_{20}f_{30}f_{02}-15g_{11}g_{20}f_{02}^2+$$
$$6f_{20}g_{20}f_{11}g_{02}-6f_{21}f_{12}+6g_{20}f_{11}^2f_{20}+66f_{40}g_{02}+27f_{30}f_{03}-45g_{05}-9g_{23}+15g_{21}f_{02}^2-$$
$$27f_{20}f_{31}-9g_{41}+3g_{21}g_{12}+9g_{11}g_{20}^2f_{11}-15f_{11}f_{02}f_{03}-45f_{50}+12f_{30}g_{11}f_{20}+10g_{20}f_{20}^3-$$
$$48f_{20}g_{20}g_{30}-10g_{02}f_{02}^2g_{11}-9g_{11}^2g_{20}f_{20}+13g_{21}f_{20}^2-9f_{14}-9f_{32}-15g_{21}g_{20}^2).$$

For the first time this result was apparently obtained by Serebryakova [Serebryakova, 1959].

To compute the third Lyapunov quantity, we obtain $\widetilde{T}_4$:

$$\widetilde{T}_4 = \frac{\pi}{1152}(784f_{20}^4+1540g_{20}^4+49g_{11}^4-352g_{21}g_{20}g_{11}+48g_{21}^2-336f_{40}g_{11}+$$
$$2616f_{20}^2f_{03}+480g_{22}g_{02}+200g_{20}^2f_{02}^2+700f_{11}g_{20}^3+270f_{03}g_{30}-154f_{20}g_{11}^3+1728g_{40}g_{02}+$$
$$424g_{11}^2g_{02}^2-48f_{22}g_{11}+54f_{03}f_{21}+453f_{02}^2g_{11}^2-2184f_{20}^2g_{30}+400g_{02}^4+864f_{30}^2+1540f_{02}^4+$$
$$556g_{20}^2f_{02}g_{11}+945g_{30}^2+240f_{12}^2+768f_{20}f_{40}+2352f_{20}f_{02}g_{02}^2-320g_{11}g_{21}g_{02}-180g_{20}^2f_{20}g_{11}-$$
$$1134f_{11}g_{30}g_{20}+5172f_{20}f_{03}f_{02}+513f_{03}^2+153g_{12}^2+f_{11}^4-762f_{11}f_{03}g_{02}-1800f_{20}^2g_{20}f_{11}-$$
$$708f_{20}g_{20}g_{11}g_{02}-888g_{12}g_{02}^2-84f_{21}g_{20}^2+1040f_{02}^2g_{02}^2+432g_{40}f_{11}+4692f_{20}^2f_{02}^2+648f_{21}g_{02}^2+$$
$$1180f_{02}^3g_{11}-96g_{21}f_{02}g_{02}-198f_{11}g_{30}g_{02}-480f_{12}g_{20}f_{02}-1500g_{12}g_{20}g_{02}+228f_{20}^2f_{02}g_{11}+$$
$$912f_{02}f_{12}f_{11}-1944g_{30}g_{02}^2+672f_{02}f_{40}-48g_{22}f_{11}+402f_{03}g_{11}^2+2772g_{20}^2g_{02}^2-63f_{20}^2g_{11}^2+$$
$$150f_{21}g_{11}^2+444f_{21}f_{02}^2+3300f_{03}f_{02}^2+384f_{22}f_{02}+432g_{11}f_{04}-712f_{11}g_{02}^3+880f_{20}f_{12}f_{11}-$$
$$18f_{11}f_{03}g_{20}+1992f_{20}f_{02}^2g_{11}+162f_{20}f_{21}g_{11}-2080f_{20}f_{12}g_{02}-150f_{11}f_{02}g_{11}g_{02}-828f_{11}g_{20}g_{02}^2-$$
$$96f_{11}f_{30}g_{11}+1392f_{11}f_{30}f_{02}+980g_{11}^2g_{20}g_{02}+64f_{11}f_{20}g_{21}-64f_{12}f_{11}g_{11}-6f_{11}f_{21}g_{20}-$$
$$1812g_{30}f_{20}f_{02}+112g_{11}f_{12}g_{02}-2744f_{11}f_{20}^2g_{02}-1280f_{20}g_{20}f_{12}+680f_{02}^2g_{20}g_{02}+80f_{12}g_{20}g_{11}-$$
$$4164f_{20}f_{02}f_{11}g_{02}-1212f_{20}f_{02}g_{12}-408f_{20}g_{11}g_{02}^2+102f_{11}g_{20}g_{12}-3552f_{20}f_{30}g_{20}+$$
$$22f_{11}^2f_{20}g_{11}-1128f_{20}^2g_{12}-48g_{20}f_{13}+696f_{11}^2f_{20}^2+672g_{04}g_{20}+2336f_{20}^2g_{02}^2-162f_{21}g_{12}+$$
$$96g_{31}g_{11}+50f_{11}^2g_{11}^2+1128f_{03}g_{02}^2-3780g_{30}g_{20}^2+21f_{11}^2g_{20}^2+480f_{20}f_{22}-432g_{02}f_{31}+$$
$$2016g_{40}g_{20}+3080g_{20}^3g_{02}+66f_{11}^2f_{03}-48f_{02}g_{31}+768g_{04}g_{02}+1728f_{04}f_{20}-54f_{11}^2g_{12}+$$
$$3984g_{20}^2f_{20}^2-2f_{11}^3g_{20}-198f_{03}g_{12}-432f_{20}g_{13}+630g_{30}g_{12}-46f_{11}f_{20}g_{11}g_{02}+10f_{11}g_{20}f_{20}g_{11}+$$
$$30f_{20}g_{12}g_{11}+318f_{20}f_{02}g_{11}^2+168f_{11}g_{20}^2g_{02}-4788g_{30}g_{20}g_{02}-2400f_{02}f_{30}g_{02}+1734f_{20}f_{03}g_{11}+$$
$$4168f_{02}f_{20}g_{20}g_{02}-62f_{02}f_{11}g_{20}g_{11}+816f_{30}g_{20}g_{11}-2260f_{11}g_{20}f_{20}f_{02}-82f_{11}g_{20}g_{11}^2+$$
$$2792g_{20}^2f_{20}f_{02}+96f_{13}f_{11}+9f_{21}^2-1536f_{02}f_{12}g_{02}+928f_{20}g_{20}g_{21}+620f_{02}g_{20}g_{11}g_{02}-$$
$$924g_{12}g_{20}^2+1424g_{20}g_{02}^3-396f_{02}^2g_{12}-546g_{30}g_{11}^2+804g_{20}^2g_{11}^2-180f_{02}^2g_{30}-2f_{02}g_{11}^3-$$
$$528f_{13}g_{02}-18g_{30}f_{11}^2+6f_{21}f_{11}^2+864f_{30}f_{12}-198g_{12}g_{11}^2+4040f_{02}^3f_{20}+369f_{11}^2g_{02}^2-$$
$$330f_{11}g_{11}^2g_{02}-1728g_{20}f_{30}f_{02}+90f_{21}f_{02}g_{11}+216f_{02}g_{11}g_{02}^2+126f_{11}^2g_{20}g_{02}-90f_{02}g_{11}g_{12}+$$
$$1392f_{20}f_{30}f_{11}+1614f_{03}f_{02}g_{11}+5488f_{20}^2g_{20}g_{02}+32f_{11}g_{21}g_{11}-294g_{30}f_{20}g_{11}+1332f_{11}^2f_{20}f_{02}-$$
$$1876f_{11}f_{02}^2g_{02}+1308f_{02}f_{20}f_{21}-3840f_{20}f_{30}g_{02}+800f_{20}g_{21}g_{02}+692f_{11}^2f_{02}^2-300f_{03}g_{20}^2-$$
$$288f_{30}g_{21}+2768f_{20}^3f_{02}-58f_{11}^3g_{02}-144g_{20}f_{31}-528g_{31}f_{20}-616f_{20}^3g_{11}+984f_{20}^2f_{21}+$$
$$384g_{20}g_{22}-144f_{02}g_{13}+90f_{21}g_{30}+2016f_{02}f_{04}-336f_{11}g_{04}-366f_{02}g_{30}g_{11}+96f_{02}f_{11}g_{21}-$$
$$18f_{02}f_{11}^2g_{11}+468f_{03}g_{20}g_{02}-96f_{02}g_{20}g_{21}+144f_{41}-720g_{50}+720f_{05}+144f_{23}-144g_{14}-$$

$144g_{32}+816f_{30}g_{11}g_{02}+510f_{11}g_{12}g_{02}+444f_{21}g_{20}g_{02}-500g_{20}f_{11}f_{02}^2-222f_{11}f_{21}g_{02})$.

To compute L_3 in the general case, applying the mentioned above algorithms, it is necessary to treat the symbolic expressions involving more than two millions of symbols. Therefore, to overcome the restrictions while using a main memory in the packets of symbolic computations, we consider the following case

$$f_{20}=f_{30}=f_{40}=f_{50}=f_{60}=f_{70}=0.$$

General system (221) is reduced to this form by the change

$$y_{old}=y_{new}+f_2x^2+f_3x^3+f_4x^4+f_5x^5+f_6x^6+f_7x^7, \tag{222}$$

where

$f_2=f_{20}$,

$f_3=f_{30}+f_{20}f_{11}$,

$f_4=f_{02}f_{20}^2+f_{40}+f_{11}f_{30}+f_{20}f_{11}^2+f_{20}f_{21}$,

$f_5=f_{20}f_{31}+f_{50}+3f_{11}f_{02}f_{20}^2+f_{11}f_{40}+f_{11}^2f_{30}+f_{20}f_{11}^3+2f_{11}f_{20}f_{21}+f_{21}f_{30}+2f_{20}f_{02}f_{30}+f_{12}f_{20}^2$,

$f_6=f_{22}f_{20}^2+f_{20}f_{41}+f_{03}f_{20}^3+f_{60}+3f_{21}f_{02}f_{20}^2+2f_{21}f_{11}f_{30}+3f_{21}f_{20}f_{11}^2+2f_{31}f_{20}f_{11}+6f_{11}^2f_{02}f_{20}^2+3f_{11}f_{12}f_{20}^2+2f_{20}f_{02}f_{40}+6f_{11}f_{20}f_{02}f_{30}+2f_{20}^3f_{02}^2+2f_{20}f_{12}f_{30}+f_{21}f_{40}+f_{20}f_{21}^2+f_{31}f_{30}+f_{11}f_{50}+f_{11}^2f_{40}+f_{11}^3f_{30}+f_{20}f_{11}^4+f_{02}f_{30}^2$,

$f_7=3f_{21}f_{11}^2f_{30}+2f_{20}f_{02}f_{50}+2f_{41}f_{20}f_{11}+4f_{21}f_{20}f_{11}^3+6f_{20}^2f_{12}f_{11}^2+3f_{11}f_{20}f_{21}^2+2f_{31}f_{20}f_{21}+3f_{02}f_{11}f_{30}^2+3f_{31}f_{20}f_{11}^2+6f_{20}^2f_{02}^2f_{30}+3f_{03}f_{20}^2f_{30}+2f_{02}f_{30}f_{40}+2f_{20}f_{22}f_{30}+2f_{11}f_{21}f_{40}+3f_{20}^2f_{02}f_{31}+3f_{20}^2f_{12}f_{21}+4f_{03}f_{20}^3f_{11}+2f_{31}f_{11}f_{30}+4f_{20}^3f_{12}f_{02}+3f_{20}^2f_{22}f_{11}+10f_{20}^3f_{02}^2f_{11}+2f_{20}f_{12}f_{40}+10f_{20}^2f_{02}f_{11}^3+f_{13}f_{20}^3+f_{21}^2f_{30}+f_{32}f_{20}^2+f_{20}f_{51}+f_{41}f_{30}+f_{12}f_{30}^2+f_{31}f_{40}+f_{11}f_{60}+f_{11}^2f_{50}+f_{11}^3f_{40}+f_{11}^4f_{30}+f_{20}f_{11}^5+f_{21}f_{50}+f_{70}+6f_{20}f_{12}f_{11}f_{30}+6f_{20}f_{02}f_{11}f_{40}+12f_{20}f_{02}f_{11}^2f_{30}+12f_{20}^2f_{02}f_{11}f_{21}+6f_{20}f_{02}f_{21}f_{30}$.

Note that this change is nonsingular and does not change the Lyapunov quantities of system since

$$y_{new}(0)=y_{old}(0)=h, y_{new}(T)=y_{old}(T).$$

For $\widetilde{T}_5$ we have

$\widetilde{T}_5=-\dfrac{\pi}{4320}(23130f_{03}f_{02}^2g_{11}+2340g_{20}^2f_{02}^2g_{11}+438f_{11}^2g_{11}g_{20}g_{02}-1110f_{11}g_{11}f_{21}g_{02}-720g_{32}g_{11}+2304f_{04}g_{11}^2-144f_{22}g_{11}^2+1200g_{11}f_{12}^2-1020g_{12}f_{02}^3+1296f_{04}f_{03}+576g_{02}^2g_{13}+576f_{04}g_{02}^2-288f_{22}g_{12}-960f_{02}^2g_{21}g_{02}-1648g_{21}g_{11}^2g_{02}-810f_{03}g_{30}f_{02}-1856g_{21}g_{20}g_{11}^2+2880f_{22}f_{02}^2+384g_{31}g_{02}^2+1296f_{03}g_{13}+960f_{22}g_{20}^2-1296g_{30}f_{04}-432g_{12}g_{13}+720f_{41}g_{11}+4725f_{02}g_{30}^2+1000f_{02}^3g_{20}^2+960f_{12}g_{20}^2g_{02}+8640g_{40}g_{02}f_{02}+16840g_{20}^3g_{11}g_{02}-8420f_{02}^3f_{11}g_{02}-1530g_{30}f_{02}^2g_{11}-20724g_{30}g_{20}^2g_{11}+144g_{12}f_{11}g_{21}+2160g_{40}g_{11}f_{11}-2286g_{12}f_{02}^2g_{11}-2160g_{02}f_{31}g_{11}+149g_{11}^5+1002f_{03}f_{02}f_{11}^2+144f_{12}f_{21}f_{11}-4056f_{02}g_{02}^2g_{12}-128f_{02}g_{11}^2f_{11}^2+12096f_{04}f_{02}g_{11}-318f_{02}f_{11}^2g_{12}+4580g_{02}f_{02}^2g_{20}g_{11}+1014f_{11}^2g_{20}f_{02}g_{02}-9026f_{11}f_{02}^2g_{11}g_{02}-2032f_{02}g_{21}g_{11}g_{02}-864f_{02}g_{13}g_{11}-954g_{12}f_{21}g_{11}+1856f_{02}g_{02}^2g_{11}^2-432f_{11}g_{30}f_{12}+2790f_{11}g_{11}g_{12}g_{02}+114f_{11}f_{21}g_{20}g_{11}+7120g_{02}^3f_{02}g_{20}+3360g_{04}f_{02}g_{20}-132g_{20}^2f_{21}g_{11}+7200f_{02}f_{03}g_{11}^2+3988g_{20}g_{11}^3g_{02}-378f_{03}g_{30}g_{11}+144f_{11}^2g_{13}+2000g_{02}^4g_{11}+960g_{31}g_{20}^2+1440f_{04}g_{20}^2+864g_{30}g_{21}g_{20}-1440f_{02}^2g_{20}g_{21}-2400f_{02}^2f_{12}g_{20}-240f_{02}f_{13}g_{20}+1920g_{20}g_{11}g_{22}+3400f_{02}^3g_{20}g_{02}-720f_{04}g_{02}f_{11}-3600g_{50}f_{02}+792f_{11}g_{20}^2g_{11}g_{02}+8640g_{40}g_{02}g_{11}+288g_{21}g_{20}g_{12}-$

$$
\begin{aligned}
&864g_{21}g_{02}^2g_{20} - 864f_{03}g_{21}g_{20} + 5040f_{02}^2f_{11}f_{12} + 432f_{11}g_{30}g_{21} - 4188f_{11}g_{02}^2g_{11}g_{20} + \\
&4740f_{03}f_{02}g_{20}g_{02} - 4290f_{11}f_{03}g_{11}g_{02} - 6540g_{12}f_{02}g_{20}g_{02} - 810g_{12}f_{21}f_{02} - 846f_{02}f_{03}g_{12} - \\
&6720f_{02}^2f_{12}g_{02} - 2640f_{13}g_{02}g_{11} + 960g_{31}g_{02}g_{20} - 144g_{13}f_{11}g_{20} - 240f_{11}g_{22}g_{11} - 290f_{11}^3g_{11}g_{02} + \\
&2364f_{21}g_{11}g_{20}g_{02} + 144g_{12}g_{02}g_{21} + 900f_{02}f_{03}g_{20}^2 + 3500f_{11}f_{02}g_{20}^3 - 2000g_{20}f_{12}f_{02}g_{11} + \\
&48g_{21}g_{02}^2f_{11} + 3344g_{20}^2f_{02}g_{11}^2 - 2500f_{02}^3f_{11}g_{20} - 58f_{11}^3g_{20}f_{02} + 480f_{02}f_{13}f_{11} + 288f_{21}g_{02}f_{12} - \\
&1440g_{02}g_{20}^2g_{21} - 2640f_{13}g_{02}f_{02} + 13860g_{02}^2f_{02}g_{20}^2 + 426f_{03}f_{11}^2g_{11} - 186f_{11}^2g_{30}g_{11} + 112f_{11}g_{21}g_{11}^2 + \\
&400g_{20}f_{12}g_{11}^2 + 10080g_{20}g_{40}g_{11} - 1302f_{21}f_{02}f_{11}g_{02} - 432f_{11}f_{03}g_{21} + 1440g_{20}^2g_{13} - 1962g_{30}g_{11}^3 + \\
&432f_{21}g_{13} - 750g_{11}^3g_{12} + 480f_{13}f_{11}g_{11} + 3884f_{11}g_{20}^3g_{11} - 240f_{02}g_{22}f_{11} - 1680g_{04}g_{11}f_{11} - \\
&1854g_{30}f_{02}f_{11}g_{02} + 48g_{02}g_{20}f_{11}g_{21} - 6006f_{11}g_{30}g_{20}g_{11} - 2160g_{02}f_{31}f_{02} - 1440g_{30}g_{11}^2f_{02} - \\
&2090f_{02}^2g_{20}f_{11}g_{11} - 3560f_{11}g_{02}^3g_{11} - 18900g_{30}f_{02}g_{20}^2 - 288f_{11}f_{12}g_{02}^2 - 3660g_{12}f_{02}g_{20}^2 + \\
&6600f_{03}f_{02}g_{02}^2 - 9720f_{02}g_{02}^2g_{30} - 318g_{12}f_{11}^2g_{11} - 3176g_{02}^3f_{02}f_{11} - 48f_{11}^2f_{12}g_{20} + 2286f_{02}g_{12}g_{30} + \\
&3732f_{03}g_{20}g_{11}g_{02} + 2400g_{22}g_{02}f_{02} - 720f_{02}f_{31}g_{20} - 4632g_{02}^2g_{11}g_{12} + 1557f_{11}^2f_{02}g_{02}^2 - \\
&720f_{31}g_{11}g_{20} + 1440g_{02}g_{20}g_{13} + 432f_{12}f_{03}f_{11} + 144g_{11}g_{31}f_{02} + 7312g_{20}g_{02}^3g_{11} + 688f_{02}f_{11}g_{21}g_{11} + \\
&11520f_{04}f_{02}^2 + 432f_{21}f_{04} + 3429f_{03}^2g_{11} + 720f_{41}f_{02} + 3840g_{04}g_{02}f_{02} - 23940g_{20}f_{02}g_{30}g_{02} + \\
&1440f_{04}g_{02}g_{20} - 5388g_{12}g_{20}^2g_{11} - 144g_{21}g_{02}f_{21} + 192g_{21}f_{11}^2g_{02} - 144f_{11}f_{04}g_{20} + 3360g_{20}g_{04}g_{11} + \\
&15400g_{20}^3f_{02}g_{02} - 144f_{12}f_{11}^2g_{02} - 288g_{12}g_{02}f_{12} - 4146f_{03}f_{02}f_{11}g_{02} - 8124g_{20}g_{11}g_{12}g_{02} + \\
&414f_{11}g_{20}g_{12}g_{11} - 96g_{31}f_{11}g_{20} + 1584f_{02}f_{22}g_{11} - 522g_{30}f_{02}f_{11}^2 + 5320f_{02}^2g_{02}^2g_{11} + 656f_{12}g_{11}^2g_{02} - \\
&240g_{20}f_{13}g_{11} - 1710f_{03}g_{12}g_{11} - 434f_{11}^3f_{02}g_{02} - 480g_{02}f_{22}f_{11} + 270f_{21}f_{02}f_{11}^2 + 14724g_{02}^2g_{20}^2g_{11} + \\
&7700f_{02}^5 - 720g_{14}f_{02} + 2000g_{02}^4f_{02} - 960g_{21}g_{20}^3 + 53f_{11}^4f_{02} + 7700g_{20}^4f_{02} + 384g_{02}^2f_{22} + \\
&720f_{23}f_{02} + 240g_{21}^2g_{11} + 18900f_{02}^3f_{03} + 3940f_{02}^3f_{11}^2 + 384f_{12}g_{02}^3 - 864g_{30}g_{31} - 864g_{30}f_{22} + \\
&333f_{02}f_{21}^2 + 1200f_{02}f_{12}^2 + 477f_{02}g_{12}^2 + 1736g_{02}^2g_{11}^3 + 720f_{23}g_{11} + 3156g_{20}^2g_{11}^3 - 432f_{04}g_{12} + \\
&864f_{03}f_{22} - 720g_{14}g_{11} + 5200f_{02}^3g_{02}^2 + 3600f_{05}g_{11} + 288g_{31}f_{21} + 96g_{31}f_{11}^2 + 909g_{12}^2g_{11} + \\
&462f_{21}g_{11}^3 - 1296g_{30}g_{13} + 4725f_{03}^2f_{02} - 432g_{21}f_{03}g_{02} + 288f_{21}f_{02}g_{11}^2 + 1854f_{02}^2f_{21}g_{11} + \\
&3870g_{12}g_{30}g_{11} - 864g_{02}f_{12}g_{30} + 3624f_{21}f_{02}g_{02}^2 - 3180f_{11}g_{02}^2f_{02}g_{20} - 48f_{11}^2g_{20}g_{21} + \\
&585f_{02}g_{20}^2f_{11}^2 + 324f_{03}g_{20}^2g_{11} - 144f_{11}f_{21}g_{21} + 246g_{20}f_{03}f_{11}g_{11} - 96f_{11}f_{22}g_{20} + 864g_{02}f_{03}f_{12} - \\
&7216f_{12}f_{02}g_{11}g_{02} - 624f_{02}f_{11}g_{11}^2g_{02} + 384g_{02}g_{20}f_{12}f_{11} + 4592f_{02}g_{20}g_{11}^2g_{02} - 126f_{21}f_{02}g_{20}f_{11} + \\
&3180f_{21}f_{02}g_{20}g_{02} + 144f_{04}f_{11}^2 + 4192f_{02}f_{12}f_{11}g_{11} - 2144g_{21}f_{02}g_{20}g_{11} - 1680g_{04}f_{02}f_{11} - \\
&330f_{02}f_{03}g_{20}f_{11} + 1800f_{11}g_{20}^2f_{02}g_{02} - 432f_{02}g_{20}f_{11}g_{11}^2 - 25332g_{30}g_{20}g_{11}g_{02} - 510f_{11}g_{11}g_{30}g_{02} + \\
&960g_{02}g_{20}f_{22} - 720g_{13}g_{02}f_{11} + 432g_{02}g_{30}g_{21} + 240f_{02}g_{21}^2 + 3180f_{21}f_{02}^3 + 576g_{31}g_{11}^2 + \\
&5f_{11}^4g_{11} + 8660g_{20}^4g_{11} + 288f_{22}f_{21} - 48f_{11}^3g_{21} - 5f_{02}g_{11}^4 + 1242f_{03}g_{11}^3 + 154f_{11}^2g_{11}^3 + \\
&144g_{13}g_{11}^2 + 45f_{21}^2g_{11} - 3600g_{11}g_{50} + 3600f_{05}f_{02} + 48f_{11}^3f_{12} - 720g_{32}f_{02} - 900g_{30}f_{02}^3 + \\
&864f_{03}g_{31} + 11200f_{02}^4g_{11} - 288g_{31}g_{12} + 720f_{02}^2g_{13} + 96f_{22}f_{11}^2 - 192g_{02}^3g_{21} + 5045f_{02}^3g_{11}^2 + \\
&1391f_{02}^2g_{11}^3 + 5589g_{30}^2g_{11} + 720g_{31}f_{02}^2 + 1854f_{03}f_{21}f_{02} + 540g_{20}^2f_{02}f_{21} + 2160f_{02}g_{40}f_{11} + \\
&1845g_{02}^2f_{11}^2g_{11} - 480g_{31}g_{02}f_{11} + 6024g_{02}^2f_{03}g_{11} + 30f_{11}^2f_{21}g_{11} - 288f_{02}g_{12}g_{11}^2 - 1170f_{11}g_{11}^3g_{02} + \\
&2400g_{02}g_{11}g_{22} + 3082f_{02}^2f_{11}^2g_{11} + 162g_{11}f_{21}g_{30} + 153f_{11}^2g_{20}^2g_{11} - 414f_{02}f_{21}g_{30} - 10104g_{30}g_{02}^2g_{11} - \\
&384f_{11}g_{20}^2g_{21} - 288g_{21}f_{21}g_{20} + 3840g_{02}g_{04}g_{11} + 1782g_{12}f_{02}f_{11}g_{02} - 5670g_{20}f_{02}g_{30}f_{11} + \\
&414f_{02}g_{12}g_{20}f_{11} + 38f_{11}^3g_{20}g_{11} + 1920f_{02}g_{22}g_{20} + 480f_{11}f_{12}g_{20}^2 + 960f_{12}g_{02}^2g_{20} - 144g_{12}f_{12}f_{11} + \\
&3240g_{02}^2g_{11}f_{21} + 10080g_{20}g_{40}f_{02} - 266g_{20}f_{11}g_{11}^3 + 558f_{03}f_{21}g_{11} - 272f_{11}f_{12}g_{11}^2).
\end{aligned}
$$

From $L_1 = L_2 = 0$ we obtain g_{03} and g_{05}

$$g_{03} = \frac{1}{3}(g_{11}g_{20} - f_{11}f_{02} - 2g_{02}f_{02} + g_{11}g_{02} - f_{12} - g_{21}),$$

$$
\begin{aligned}
g_{05} = \frac{1}{45}(&6g_{20}f_{02}f_{21} + 2g_{11}^3g_{20} - 9g_{02}^2g_{21} + 9g_{21}g_{30} + 9g_{21}f_{03} - 6f_{13}g_{11} + 15g_{20}g_{31} + \\
&27g_{02}g_{13} + 9g_{13}g_{20} - 6g_{22}f_{02} + 18g_{02}^2f_{12} - 6f_{21}f_{12} + 12f_{22}g_{02} + 9g_{11}g_{40} + 21g_{02}g_{31} + \\
&15g_{11}g_{20}^3 - 6g_{12}f_{12} + 3g_{11}g_{22} - 7g_{11}^2g_{20}f_{02} - 15f_{13}f_{02} - 9g_{21}g_{20}f_{11} + 6g_{20}f_{02}g_{02}f_{11} + \\
&9g_{20}g_{11}g_{02}f_{11} + 3g_{02}f_{11}f_{12} - 15g_{20}^2g_{21} + 9g_{11}g_{20}^2f_{11} - 6g_{12}g_{11}g_{20} + 7g_{11}g_{21}f_{02} - 5g_{11}f_{02}f_{12} -
\end{aligned}
$$

$21g_{11}g_{02}g_{30} - 8g_{11}^2 g_{02}f_{02} - 3g_{20}g_{11}f_{21} + 24g_{11}g_{20}^2 g_{02} - 30g_{02}f_{02}f_{03} - 15g_{11}g_{20}f_{02}^2 -$
$24g_{11}g_{20}g_{30} + g_{11}^2 f_{11}f_{02} + 18f_{11}f_{02}g_{02}^2 + 6g_{20}f_{02}g_{12} - 5g_{11}f_{02}^2 f_{11} + 18g_{12}g_{02}f_{02} - 9f_{11}f_{02}f_{21} -$
$15f_{11}f_{02}f_{03} + 3f_{02}g_{20}f_{11}^2 + 3g_{02}f_{02}f_{11}^2 - 9g_{11}g_{20}f_{03} + 9g_{11}g_{02}^2 g_{20} - 12g_{11}f_{03}g_{02} - 10g_{11}g_{02}f_{02}^2 +$
$6g_{02}g_{20}f_{12} + 12g_{02}f_{02}f_{21} - 9g_{21}g_{02}f_{11} - 6f_{11}f_{02}g_{12} - 9g_{12}g_{11}g_{02} - 6g_{11}f_{03}f_{11} - 6g_{30}g_{11}f_{11} +$
$3g_{20}f_{12}f_{11} - 24g_{21}g_{20}g_{02} - 9f_{04}f_{11} + 3g_{12}g_{21} + 6g_{20}f_{22} + 6f_{11}g_{31} - 9f_{32} - 9g_{41} -$
$9f_{14} - 9g_{23} - 2g_{11}^2 g_{21} + 2g_{11}^3 g_{02} - 9f_{31}f_{02} - 18g_{02}f_{04} - 15g_{11}g_{04} - 3f_{02}f_{11}^3 + g_{11}^2 f_{12} -$
$3f_{22}f_{11} + 15g_{21}f_{02}^2 + 3g_{21}f_{21} - 3f_{12}f_{11}^2 - 60f_{02}g_{04}).$

Then

$L_3 = \dfrac{\pi}{1728}(30g_{02}g_{11}^3 g_{30} + 6g_{21}^3 + 36f_{02}f_{11}g_{12}g_{20}^2 - 1080g_{02}^2 g_{11}g_{20}^3 - 585g_{11}f_{11}g_{20}^4 -$
$140g_{02}g_{11}f_{02}^4 + 99f_{12}g_{02}f_{11}g_{30} + 1278g_{20}^2 g_{11}f_{11}g_{30} + 1575g_{02}^2 g_{20}g_{11}g_{30} - 99f_{02}g_{20}f_{11}f_{31} -$
$198f_{21}g_{20}^2 g_{02}f_{02} - 90f_{12}g_{02}g_{20}^3 + 144g_{30}f_{21}g_{20}f_{02} + 540g_{21}g_{20}^4 - 54f_{12}g_{12}^2 + 216g_{21}g_{30}^2 +$
$630f_{02}g_{11}^2 g_{20}^3 + 189g_{20}f_{11}g_{02}g_{11}f_{02}^2 - 108f_{12}g_{02}f_{11}g_{20}^2 + 135g_{02}^3 g_{11}g_{30} - 261f_{02}g_{02}f_{11}f_{31} +$
$36g_{20}g_{04}g_{21} + 72g_{32}g_{20}g_{11} + 126g_{21}g_{20}g_{22} + 306g_{12}f_{11}g_{20}g_{02}f_{02} + 18g_{02}g_{20}f_{11}f_{22} +$
$9g_{21}f_{21}^2 + 4g_{02}g_{11}^5 + 4g_{20}g_{11}^5 - 60g_{20}^3 g_{11}^3 - 4g_{21}g_{11}^4 + 210g_{21}f_{02}^4 - 576f_{03}g_{30}g_{20}g_{11} +$
$27g_{21}g_{12}^2 - 378f_{12}g_{02}^2 f_{03} - 6g_{11}^3 g_{20}f_{21} - 639g_{02}^2 g_{21}g_{30} - 9f_{11}^2 f_{12}g_{30} + 384f_{02}f_{12}g_{02}g_{21} +$
$6g_{21}g_{11}^2 f_{21} + 9f_{12}f_{11}^4 + 24g_{11}^3 g_{20}g_{30} - 540g_{11}g_{20}^5 + 405g_{30}g_{40}g_{11} - 144f_{11}f_{03}g_{11}g_{30} +$
$9f_{02}f_{11}^5 + 2f_{12}g_{11}^4 + 63f_{02}g_{02}^2 g_{20}f_{11}^2 - 63f_{02}g_{02}g_{12}f_{11}^2 - 90f_{11}g_{30}g_{11}g_{12} - 54g_{12}^2 f_{11}f_{02} -$
$27g_{12}f_{32} - 105f_{11}f_{02}^3 f_{21} - 420f_{02}^2 f_{03}g_{11}g_{02} + 234g_{30}f_{02}g_{02}g_{12} + 261g_{20}^3 g_{11}f_{21} + 270g_{02}^3 g_{20}g_{21} -$
$54g_{40}f_{11}g_{02}f_{02} + 42g_{20}g_{11}^2 f_{11}g_{21} + 234f_{12}g_{02}^2 g_{30} - 504f_{03}g_{21}g_{20}g_{02} + 54f_{02}f_{11}^3 f_{21} +$
$84f_{03}g_{20}g_{11}^2 f_{02} - 180g_{51}f_{11} + 210g_{02}f_{11}g_{20}f_{02}^3 - 45f_{23}g_{20}g_{11} - 180g_{04}f_{11}g_{20}f_{02} - 81g_{13}g_{22} +$
$81f_{22}f_{31} - 360g_{02}^2 g_{21}f_{02}^2 - 81g_{22}g_{31} - 84f_{02}g_{02}^2 f_{11}g_{11}^2 - 204f_{02}f_{11}g_{30}g_{11}^2 - 18g_{20}f_{12}f_{11}^3 -$
$30f_{12}f_{21}g_{11}^2 + 240g_{02}f_{02}^2 g_{11}f_{21} + 315g_{13}f_{12}f_{02} - 216f_{03}g_{41} + 63f_{02}g_{02}g_{20}f_{11}^3 + 180g_{30}g_{20}g_{11}f_{11}^2 +$
$420f_{11}g_{20}^2 g_{11}f_{02}^2 + 216f_{02}g_{11}g_{20}g_{13} - 81g_{12}g_{23} - 237f_{02}g_{21}g_{11}f_{11}g_{20} - 105f_{02}^2 f_{12}g_{11}^2 -$
$72g_{20}f_{12}f_{11}f_{21} + 18f_{02}g_{02}^2 g_{20}f_{21} - 135g_{33}g_{20} + 495g_{11}g_{50}g_{02} + 135g_{30}g_{21}g_{12} + 168g_{12}g_{02}g_{11}f_{02}^2 +$
$36f_{13}f_{21}g_{11} + 342f_{02}g_{02}g_{20}f_{31} + 54g_{31}f_{21}f_{11} + 99f_{11}^2 g_{30}g_{02}f_{02} + 102f_{02}g_{21}^2 g_{02} - 54f_{11}^2 g_{40}g_{11} -$
$90g_{20}g_{21}f_{11}f_{21} - 18f_{02}g_{02}g_{12}f_{21} + 90f_{23}f_{12} + 210f_{02}f_{12}^2 g_{02} + 210f_{22}f_{02}^2 g_{20} - 180g_{14}g_{20}f_{02} -$
$18g_{11}^2 g_{20}g_{13} - 639g_{21}g_{02}f_{11}g_{30} + 369f_{02}g_{21}g_{11}g_{30} - 210f_{02}^3 f_{13} + 45g_{14}g_{21} - 840g_{04}f_{02}^3 +$
$72f_{02}f_{11}f_{21}^2 + 315f_{03}g_{20}g_{11}f_{02}^2 - 225g_{14}g_{11}g_{02} - 234f_{13}g_{02}f_{11}f_{02} - 99g_{11}g_{20}^2 f_{31} - 135g_{50}g_{21} +$
$135g_{20}^2 f_{32} - 306g_{21}g_{12}g_{20}^2 - 9g_{20}g_{11}f_{21}^2 + 144f_{21}g_{21}f_{02}^2 - 12g_{20}f_{11}g_{02}g_{11}^3 + 9g_{20}f_{12}f_{11}g_{11}^2 -$
$81f_{32}g_{30} + 189f_{04}f_{13} - 84f_{12}g_{02}^2 g_{11}^2 + 105f_{02}f_{03}f_{12}g_{11} + 315g_{11}g_{02}^2 f_{11}g_{30} + 228g_{20}g_{11}^2 g_{02}g_{21} -$
$27f_{41}g_{21} + 72g_{02}^2 g_{21}g_{11}^2 - 207g_{20}f_{03}f_{11}g_{21} - 51f_{02}^2 g_{02}f_{11}g_{21} - 70g_{11}f_{11}f_{02}^4 - 455g_{20}g_{11}^2 f_{02}^3 +$
$105f_{11}^2 g_{20}f_{02}^3 + 455g_{21}g_{11}f_{02}^3 + 288g_{22}g_{02}g_{20}f_{02} + 36f_{12}f_{21}^2 - 135g_{60}g_{11} + 27f_{22}g_{20}g_{11}^2 +$
$9f_{32}f_{11}^2 - 54g_{30}g_{21}g_{11}^2 - 15f_{11}^2 f_{12}g_{11}^2 + 162g_{11}g_{20}f_{11}^2 f_{02}^2 + 360g_{02}^2 g_{20}g_{11}f_{02}^2 - 180f_{13}g_{02}^2 g_{11} +$
$423g_{02}f_{03}g_{31} - 423g_{02}f_{03}g_{11}g_{30} - 345f_{02}^2 g_{11}f_{11}g_{30} - 270g_{02}^3 g_{11}g_{20}^2 - 90f_{11}^2 f_{21}g_{20}f_{02} +$
$21f_{12}f_{02}g_{11}^3 - 9f_{22}g_{12}f_{11} - 132g_{02}g_{11}^3 g_{20}^2 + 18f_{02}f_{11}^3 g_{20}^2 - 420g_{20}g_{21}f_{11}f_{02}^2 - 540g_{31}g_{20}^3 +$
$54g_{41}f_{11}^2 + 180g_{24}f_{02} - 885g_{30}g_{11}^2 g_{02}f_{02} + 105f_{22}f_{02}^2 f_{11} + 63g_{21}g_{02}f_{13} + 1890f_{02}g_{06} +$
$105g_{20}f_{12}f_{11}f_{02}^2 + 585g_{21}f_{11}g_{20}^3 - 135f_{31}g_{20}f_{12} + 72g_{12}f_{11}g_{31} + 108f_{21}g_{40}g_{11} - 414f_{41}f_{02}g_{02} +$
$18f_{11}f_{03}g_{11}g_{12} - 54g_{30}f_{02}g_{22} - 54f_{21}g_{40}f_{02} - 48f_{03}g_{21}g_{11}^2 - 156f_{02}g_{21}g_{11}f_{11}g_{02} + 18g_{02}g_{11}f_{11}^2 f_{02}^2 -$
$288g_{20}^2 g_{02}^2 f_{11}f_{02} + 9f_{23}g_{21} + 135f_{06}f_{11} - 270g_{02}^3 g_{20}f_{12} - 1005g_{02}g_{30}g_{11}f_{02}^2 + 108g_{30}g_{11}g_{22} -$
$81f_{14}f_{11}g_{02} - 90f_{23}f_{02}g_{20} + 81f_{11}g_{20}g_{23} + 144f_{02}g_{02}f_{11}g_{20}g_{30} + 1080g_{21}g_{02}f_{11}g_{20}^2 +$
$18g_{12}f_{13}g_{11} - 504g_{02}^2 g_{21}f_{02}g_{11} + 405g_{23}g_{02}^2 - 315f_{02}^3 f_{31} - 45f_{21}g_{02}f_{11}g_{21} + 54g_{11}g_{12}g_{22} -$
$90g_{12}g_{20}^3 f_{02} - 9f_{11}^2 g_{22}f_{02} + 81f_{22}f_{13} - 6f_{02}f_{22}g_{21} + 210g_{12}f_{02}^3 g_{20} + 522g_{41}g_{02}^2 + 315f_{03}^2 f_{11}f_{02} +$
$450g_{50}g_{20}g_{11} - 288g_{12}g_{20}^2 g_{02}f_{02} - 315g_{20}g_{51} - 54f_{03}g_{02}g_{11}f_{11}^2 + 396f_{11}g_{20}g_{41} + 972g_{02}g_{20}g_{41} +$
$63f_{12}g_{02}g_{20}f_{11}^2 + 48f_{03}g_{20}g_{11}^3 + 12g_{21}g_{02}f_{11}g_{11}^2 + 54g_{11}f_{03}f_{31} - 198g_{02}f_{21}^2 f_{02} + 6f_{02}g_{21}g_{11}f_{11}^2 +$
$81f_{04}f_{31} + 9f_{14}g_{11}^2 - 30f_{02}g_{21}^2 f_{11} + 450g_{41}g_{20}^2 + 3f_{22}g_{11}g_{21} - 210f_{02}^3 g_{22} + 345g_{31}f_{02}^2 f_{11} -$

$108g_{12}f_{02}g_{22} + 90g_{20}^2f_{21}g_{11}f_{11} + 99g_{20}g_{21}f_{31} - 9f_{12}g_{02}^2f_{11}^2 + 108g_{12}^2f_{02}g_{20} - 36f_{11}g_{22}g_{20}f_{02} -$
$969g_{30}g_{20}g_{11}^2f_{02} - 135g_{02}^2g_{11}g_{22} - 45f_{32}g_{11}^2 - 288g_{22}g_{02}g_{20}g_{11} - 54f_{13}g_{02}f_{11}g_{11} - 495g_{51}g_{02} -$
$210f_{02}^2g_{11}f_{11}f_{03} + 54g_{30}g_{13}f_{11} - 315f_{02}^2f_{32} - 27f_{31}g_{12}f_{02} + 216g_{30}g_{20}g_{13} + 36f_{12}g_{30}g_{11}^2 -$
$63f_{12}g_{02}f_{11}g_{12} - 135g_{20}^3g_{13} - 315f_{02}^2g_{23} + 54f_{33}g_{11} + 3105g_{02}g_{20}^2g_{30}g_{11} + 756f_{04}g_{04} +$
$177f_{12}f_{02}g_{11}f_{11}g_{20} - 243g_{02}g_{30}g_{11}f_{21} - 45f_{11}f_{02}g_{11}^2f_{21} + 171f_{13}f_{21}f_{02} - 90f_{13}f_{11}g_{20}f_{02} -$
$54f_{11}g_{30}f_{21}g_{11} - 54g_{33}f_{11} - 315g_{31}g_{02}^2f_{11} + 243g_{31}f_{21}g_{02} + 30f_{02}g_{21}^2g_{20} + 153f_{13}f_{11}^2f_{02} +$
$18g_{11}^3g_{20}g_{12} - 135f_{05}g_{21} + 66f_{21}f_{02}g_{02}g_{11}^2 + 234f_{02}g_{02}f_{11}g_{20}f_{21} + 351g_{30}g_{02}g_{13} + 117g_{04}f_{11}^2f_{02} -$
$504g_{04}f_{02}g_{11}^2 - 45f_{42}f_{11} + 504f_{04}f_{02}g_{20}g_{11} - 1140g_{21}g_{02}g_{20}f_{02}^2 + 63f_{02}f_{14}g_{11} + 18g_{13}g_{11}f_{12} -$
$54f_{11}^4g_{02}f_{02} + 144g_{02}g_{20}g_{30}f_{12} - 81g_{04}g_{31} - 27f_{02}f_{22}g_{11}f_{11} + 63f_{04}g_{11}f_{12} - 63f_{02}^2g_{11}g_{22} -$
$270f_{11}g_{11}g_{30}^2 + 270f_{11}g_{30}g_{31} - 246g_{20}f_{02}g_{02}f_{12}g_{11} + 108f_{41}f_{12} - 45g_{12}g_{20}g_{11}f_{21} +$
$171f_{03}f_{11}f_{02}f_{21} + 81f_{03}g_{21}f_{21} + 180g_{20}^2g_{11}f_{11}g_{12} + 405g_{02}^2g_{20}g_{11}g_{12} + 180g_{11}g_{50}f_{11} +$
$54f_{11}f_{03}g_{13} - 252f_{02}g_{11}f_{32} - 108f_{11}^2g_{20}g_{31} - 90g_{20}^3f_{21}f_{02} + 225g_{20}f_{11}g_{02}g_{11}g_{12} - 12f_{12}f_{02}g_{11}f_{11}^2 +$
$630f_{03}^2g_{02}f_{02} - 9f_{02}f_{11}^3g_{12} + 18f_{02}f_{11}^2g_{20}g_{12} - 30f_{12}^2g_{21} - 30f_{13}g_{11}^3 + 54g_{04}f_{22} - 531g_{02}g_{30}g_{11}g_{12} -$
$216f_{03}g_{02}g_{11}g_{12} + 45g_{24}g_{11} - 660f_{02}g_{21}g_{11}g_{20}^2 + 18g_{20}f_{12}f_{11}g_{12} - 126g_{12}g_{20}g_{11}f_{02}^2 -$
$189g_{40}g_{31} - 90g_{20}^3f_{22} + 81f_{21}f_{04}f_{11} - 27g_{32}g_{21} + 351g_{20}^3g_{11}g_{12} + 420f_{02}^2f_{12}f_{11}g_{02} -$
$48f_{02}g_{21}g_{11}^3 + 135f_{02}f_{33} + 54g_{13}f_{13} - 18g_{20}g_{11}f_{11}f_{31} + 180f_{02}g_{02}g_{30}f_{21} + 252g_{02}g_{11}f_{03}^2 -$
$15f_{02}f_{11}^3g_{11}^2 - 105f_{11}f_{02}^3g_{11}^2 - 198f_{02}g_{02}f_{03}f_{11}^2 - 396g_{21}g_{02}g_{20}f_{21} + 27f_{02}g_{02}^2f_{11}f_{21} +$
$72f_{03}g_{20}g_{11}g_{02}f_{11} + 6g_{22}g_{11}^3 + 315f_{03}f_{13}f_{02} + 27f_{21}f_{32} - 54g_{31}f_{31} - 351g_{33}g_{02} - 351g_{30}g_{20}g_{11}f_{21} +$
$270g_{02}^3f_{22} + 135g_{21}g_{02}g_{20}f_{11}^2 + 72g_{02}f_{03}f_{22} + 36f_{11}f_{03}f_{22} - 54f_{24}g_{20} - 414f_{42}g_{02} -$
$45f_{02}f_{11}g_{30}f_{21} - 180g_{02}^2f_{03}g_{11}f_{11} - 90f_{13}g_{02}g_{20}f_{02} + 27f_{24}f_{11} - 675g_{15}g_{02} + 81g_{30}g_{04}g_{11} -$
$18f_{22}g_{12}g_{02} + 48g_{20}f_{02}g_{11}^4 + 54g_{32}f_{12} + 135g_{11}g_{12}g_{40} - 54g_{02}f_{11}^3f_{12} - 108g_{12}g_{41} +$
$540f_{02}g_{02}^2f_{11}g_{12} - 18f_{02}f_{11}g_{30}g_{12} + 42f_{03}f_{12}g_{11}^2 - 180g_{21}g_{02}f_{11}g_{12} - 135g_{31}g_{02}^3 - 45f_{11}^2g_{20}f_{03}f_{02} -$
$144g_{02}f_{03}f_{21}g_{11} + 27f_{04}f_{11}^3 - 270g_{20}f_{11}g_{02}^3f_{02} - 54f_{12}g_{30}g_{12} + 174f_{02}f_{22}g_{20}g_{11} + 228g_{11}g_{02}^2f_{11}f_{02}^2 -$
$72f_{22}f_{21}g_{20} - 24g_{21}g_{12}g_{11}^2 - 81g_{30}g_{23} + 216f_{03}g_{21}g_{30} - 270f_{14}g_{02}^2 - 207f_{12}g_{02}f_{11}f_{21} +$
$21f_{02}^2f_{11}g_{11}^3 - 48f_{02}^2g_{11}f_{11}^3 - 162g_{04}g_{02}g_{21} + 360f_{02}g_{40}g_{11}^2 - 70f_{12}g_{11}f_{02}^3 + 135f_{51}f_{02} -$
$81g_{40}g_{13} - 135g_{20}g_{15} + 54g_{12}f_{14} + 126f_{03}f_{11}f_{02}g_{12} - 6f_{02}f_{11}g_{12}g_{11}^2 - 216g_{30}g_{41} +$
$750g_{20}^3g_{11}f_{02}^2 + 792f_{02}g_{11}g_{02}g_{13} - 540g_{20}^2f_{11}g_{31} + 270f_{12}f_{11}g_{02}^3 + 405g_{12}g_{02}g_{13} + 117g_{32}g_{11}g_{02} -$
$195f_{02}^2g_{11}f_{11}f_{21} + 216g_{31}f_{21}g_{20} - 54g_{40}f_{12}g_{02} + 36f_{12}g_{02}^2f_{21} + 63f_{12}g_{02}^2f_{11}g_{20} - 270f_{02}g_{02}^2g_{20}g_{12} -$
$9f_{11}g_{22}f_{12} - 135g_{02}^2g_{21}f_{21} + 504f_{03}g_{04}g_{11} + 45g_{31}f_{12}f_{02} - 84f_{03}g_{21}f_{02}g_{11} - 108f_{03}g_{21}f_{11}g_{02} -$
$54g_{20}f_{04}f_{11}g_{02} - 135g_{13}f_{11}g_{20}g_{02} - 33g_{11}f_{12}^2g_{02} + 270f_{06}g_{02} - 102g_{20}g_{21}^2g_{11} - 216f_{21}f_{04}g_{02} +$
$54g_{42}f_{02} + 27f_{14}f_{11}^2 - 63g_{04}f_{11}g_{20}g_{11} - 420g_{02}f_{03}f_{02}^3 - 210f_{03}f_{11}f_{02}^3 - 900g_{14}f_{02}g_{02} +$
$54f_{21}f_{14} - 210g_{20}g_{11}f_{02}^4 + 216g_{02}^2g_{20}f_{03}g_{11} - 891g_{20}g_{11}g_{30}^2 + 945g_{06}g_{11} - 189g_{04}g_{13} +$
$54f_{04}g_{22} - 216f_{24}g_{02} - 135f_{03}f_{32} + 33f_{12}g_{02}f_{11}g_{11}^2 - 231f_{12}f_{02}g_{11}g_{12} + 315f_{02}f_{15} -$
$450g_{41}f_{02}^2 + 180f_{15}g_{11} - 84g_{04}g_{11}^3 - 27f_{21}g_{23} - 108f_{21}g_{41} + 54f_{11}f_{03}f_{21}g_{11} + 45f_{11}^2f_{12}f_{21} +$
$171f_{32}g_{02}^2 + 396f_{12}g_{02}g_{20}g_{12} + 24g_{20}f_{02}^2g_{11}f_{21} + 18f_{23}g_{11}f_{11} - 54g_{40}f_{22} - 27g_{42}g_{11} +$
$630f_{05}f_{02}g_{02} - 198g_{02}g_{20}^2f_{22} - 504g_{40}f_{11}g_{02}g_{11} - 81g_{20}g_{11}g_{12}^2 + 504f_{02}g_{02}^2g_{20}g_{11}^2 -$
$18f_{22}g_{20}f_{11}^2 - 42g_{31}g_{20}g_{11}^2 + 210f_{12}f_{22}f_{02} + 315f_{02}^2g_{20}g_{13} - 90g_{02}f_{03}g_{20}f_{11}f_{02} - 36g_{11}^2g_{23} -$
$135f_{03}g_{23} + 81f_{34} + 135f_{16} + 135g_{61} + 135f_{52} + 135g_{25} + 81g_{43} + 945g_{07} + 27g_{13}f_{21}g_{20} -$
$45g_{30}f_{02}f_{12}g_{11} + 135g_{30}g_{21}f_{21} + 180f_{12}g_{14} - 180g_{20}f_{42} + 1350g_{21}g_{02}g_{20}^3 + 111g_{21}g_{20}g_{11}f_{12} +$
$495g_{02}^2g_{21}f_{11}g_{20} - 954g_{40}g_{20}^2g_{11} - 360f_{03}g_{20}^2g_{21} + 57f_{12}f_{22}g_{11} - 1260g_{04}f_{02}^2g_{11} - 9f_{13}g_{20}^2g_{11} +$
$216g_{11}f_{03}g_{40} - 36g_{04}g_{20}^2g_{11} + 420g_{02}^2f_{11}f_{02}^3 + 126f_{04}f_{02}g_{11}f_{11} + 1140g_{02}g_{20}^2g_{11}f_{02}^2 +$
$267g_{20}^2f_{11}g_{11}^2f_{02} + 210f_{12}g_{02}g_{20}f_{02}^2 - 96g_{21}^2g_{11}g_{02} + 15f_{12}^2g_{20}g_{11} - 90g_{32}f_{02}g_{20} + 63g_{04}f_{11}g_{21} -$
$18g_{11}^2g_{41} - 27g_{20}f_{14}f_{11} - 360f_{02}g_{41}g_{11} - 45f_{12}f_{11}g_{20}^3 - 72f_{21}^2g_{20}f_{02} - 90f_{02}g_{02}f_{11}g_{20}^3 -$
$126g_{04}g_{12}f_{02} - 171g_{22}g_{20}^2g_{11} - 63f_{03}f_{12}f_{11}g_{02} + 420g_{02}f_{02}^3g_{12} - 1800g_{21}g_{02}g_{20}g_{30} +$
$234f_{02}g_{02}^2f_{11}g_{30} + 1548g_{20}f_{11}g_{02}g_{11}g_{30} + 189g_{21}g_{12}f_{02}^2 - 1026g_{04}g_{02}f_{11}f_{02} - 117f_{03}g_{20}g_{11}f_{21} +$

$$135g_{20}^2g_{23} + 18g_{40}g_{11}^3 - 93g_{21}f_{11}^2f_{02}^2 + 105f_{11}^2f_{12}f_{02}^2 + 450g_{30}g_{21}f_{02}^2 + 54f_{31}f_{12}f_{11} -$$
$$198g_{22}f_{11}g_{02}f_{02} - 168g_{02}g_{11}^3f_{02}^2 + 9f_{11}^3f_{22} + 81f_{03}g_{21}g_{12} + 90f_{12}g_{20}^2f_{21} + 9f_{11}g_{30}f_{22} -$$
$$1350g_{11}g_{02}g_{20}^4 - 1236g_{02}g_{20}f_{02}g_{11}g_{21} + 63f_{03}f_{12}f_{11}^2 - 750g_{20}^2g_{21}f_{02}^2 - 1494g_{20}g_{11}g_{02}g_{40} +$$
$$90f_{12}g_{12}g_{20}^2 - 12g_{20}^2g_{11}^3f_{11} - 135f_{11}^2g_{20}^2g_{11}g_{02} + 63f_{41}f_{02}f_{11} + 213f_{12}g_{02}g_{11}g_{21} - 90g_{14}g_{20}g_{11} +$$
$$126f_{03}^2g_{11}f_{11} - 315f_{03}g_{21}f_{02}^2 - 162f_{03}g_{20}g_{11}g_{12} - 234g_{32}f_{02}g_{02} + 540f_{12}g_{02}^2g_{12} + 210g_{02}f_{02}^3f_{11}^2 -$$
$$270f_{04}g_{12}g_{02} + 135g_{20}^2f_{02}f_{31} - 180g_{04}g_{20}f_{12} - 18f_{21}f_{02}g_{22} - 432g_{02}g_{20}^2g_{13} + 1260f_{03}g_{04}f_{02} -$$
$$180f_{41}f_{02}g_{20} + 144f_{04}g_{11}^2g_{02} - 54f_{21}f_{04}g_{20} - 123f_{02}f_{11}g_{21}f_{12} + 6f_{11}g_{21}g_{11}f_{12} - 27g_{20}f_{04}f_{11}^2 -$$
$$171f_{04}g_{21}g_{11} - 15f_{12}f_{02}g_{11}f_{11}g_{02} - 756g_{02}g_{11}g_{30}^2 + 147g_{21}g_{11}^2f_{02}^2 - 1080f_{11}g_{02}g_{11}g_{20}^3 -$$
$$468g_{30}g_{20}g_{11}g_{12} + 1005g_{31}f_{02}^2g_{02} - 522g_{40}g_{02}^2g_{11} + 378f_{04}f_{03}g_{02} - 54g_{20}f_{14}g_{02} + 360f_{03}g_{31}g_{20} -$$
$$54g_{12}f_{02}g_{40} + 117g_{04}f_{11}f_{12} - 315f_{02}^2f_{13}g_{11} + 81g_{21}g_{13}f_{02} - 30f_{12}f_{02}g_{11}g_{20}^2 - 147g_{11}^3g_{20}f_{02}^2 +$$
$$270f_{02}g_{02}^3f_{11}^2 - 54f_{12}g_{30}f_{21} - 495g_{11}g_{02}^2f_{11}g_{20}^2 + 108f_{03}f_{02}f_{11}^3 - 9f_{02}g_{02}^2f_{11}^3 + 156g_{20}^2g_{21}g_{11}^2 -$$
$$261f_{11}^2f_{21}g_{02}f_{02} - 30g_{21}^2g_{11}f_{11} + 72f_{02}f_{11}^2g_{20}g_{30} - 360f_{23}f_{02}g_{02} - 45g_{02}g_{11}f_{11}g_{22} -$$
$$350g_{02}f_{02}^3g_{11}^2 - 6f_{12}g_{12}g_{11}^2 + 261g_{12}g_{20}g_{31} - 99g_{02}f_{12}f_{13} - 84f_{02}g_{02}f_{03}g_{11}^2 + 450f_{02}^2g_{11}g_{40} -$$
$$288g_{22}g_{02}f_{12} - 54g_{02}f_{11}^2f_{22} - 153f_{11}g_{20}g_{12}g_{21} - 147f_{03}f_{11}f_{02}g_{11}^2 + 207f_{03}g_{20}^2g_{11}f_{11} +$$
$$45g_{22}g_{21}f_{11} + 675g_{30}g_{20}g_{31} + 162g_{04}g_{02}g_{20}g_{11} + 144g_{12}f_{02}g_{20}g_{30} + 204f_{02}g_{11}f_{11}g_{31} -$$
$$450f_{13}g_{02}^2f_{02} + 750f_{02}^2g_{20}g_{31} - 189f_{02}f_{13}g_{11}^2 - 90g_{20}g_{22}f_{12} + 756g_{30}g_{02}g_{31} + 117g_{21}g_{02}f_{31} -$$
$$45f_{02}f_{31}g_{11}^2 + 135f_{23}f_{02}f_{11} + 18g_{12}g_{20}f_{22} - 1026g_{04}g_{02}f_{12} + 12f_{11}g_{11}^2g_{31} - 36g_{31}g_{11}f_{12} -$$
$$27g_{40}f_{12}f_{11} + 63f_{11}^2f_{31}f_{02} + 210f_{02}^3g_{20}f_{21} + 414g_{12}g_{02}g_{31} + 18g_{20}g_{02}^2f_{22} - 1305g_{02}g_{20}^2g_{31} +$$
$$144g_{30}g_{20}f_{22} + 144g_{22}g_{02}g_{21} + 432g_{02}g_{20}g_{23} + 36f_{21}f_{02}g_{20}g_{12} + 609f_{02}g_{31}g_{20}g_{11} +$$
$$18g_{32}g_{11}f_{11} + 171g_{02}^2f_{31}f_{02} + 630f_{02}^2g_{02}g_{13} + 162g_{12}g_{13}g_{20} + 9g_{20}f_{13}g_{21} + 90g_{22}g_{20}^2f_{02} -$$
$$90g_{02}f_{11}^2g_{31} - 9f_{22}f_{11}g_{02}^2 + 315f_{02}^2g_{13}f_{11} + 216f_{12}g_{02}g_{20}f_{21} - 24g_{21}^2f_{12} + 180g_{30}g_{02}f_{22} +$$
$$171f_{04}g_{11}^2g_{20} + 270g_{22}g_{02}^2f_{02} - 30f_{03}g_{11}^3f_{11} + 144f_{11}f_{03}g_{31} + 315f_{05}f_{02}f_{11} - 405g_{20}g_{02}^2g_{13} +$$
$$36f_{22}f_{21}f_{11} + 126f_{03}f_{13}g_{11} + 360g_{02}g_{11}f_{05} - 612g_{02}g_{20}g_{12}g_{21} - 378f_{02}f_{03}g_{02}g_{12} + 33g_{02}f_{11}^2f_{02}g_{11}^2 -$$
$$111f_{12}f_{02}g_{11}f_{21} + 126g_{12}f_{13}f_{02} + 108f_{21}f_{02}f_{31} - 81f_{31}g_{30}f_{02} + 135f_{05}g_{20}g_{11} - 9g_{31}g_{21}f_{02} +$$
$$36g_{11}f_{12}^2f_{11} + 36f_{11}g_{20}f_{32} - 144f_{23}g_{11}g_{02} + 240f_{02}g_{11}g_{02}f_{22} - 288g_{02}f_{03}f_{21}f_{02} - 738f_{11}g_{20}g_{30}g_{21} -$$
$$135g_{02}g_{11}g_{12}^2 + 396g_{11}g_{02}g_{20}^2f_{21} - 270g_{02}^2g_{21}g_{12} - 9f_{11}^2f_{12}g_{12} - 828f_{03}f_{02}g_{02}^2f_{11} - 936g_{20}g_{02}^2g_{31} -$$
$$45f_{13}g_{20}f_{12} + 54g_{04}f_{21}f_{02} + 45g_{20}f_{11}g_{02}g_{11}f_{21} + 46g_{02}f_{02}g_{11}^4 - 348f_{02}g_{02}g_{12}g_{11}^2 + 156f_{02}g_{02}f_{11}g_{20}g_{11}^2 +$$
$$180f_{05}g_{11}f_{11} + 135f_{11}g_{02}g_{23} - 297f_{02}g_{11}g_{23} + 522g_{40}g_{21}g_{02} - 30g_{31}g_{11}^2g_{02} - 108g_{02}f_{02}g_{20}^2f_{11}^2 -$$
$$270f_{04}g_{02}^2f_{11} + 18g_{32}f_{02}f_{11} + 54f_{31}g_{30}g_{11} + 504g_{40}g_{20}g_{21} + 18f_{31}g_{21}f_{11} - 9f_{02}f_{11}^3g_{30} +$$
$$135g_{20}g_{02}^2g_{11}f_{21} - 213f_{12}g_{02}g_{11}^2g_{20} + 54g_{11}^2g_{02}g_{13} + 54g_{13}g_{11}g_{21} - 252f_{31}f_{02}^2g_{11} - 18f_{12}g_{12}f_{21} -$$
$$18f_{02}g_{20}f_{11}^4 + 30g_{20}f_{12}f_{02}g_{21} + 108f_{11}f_{12}f_{13} + 18f_{03}f_{11}^3g_{11} - 1200g_{20}f_{02}^2g_{30}g_{11} - 54f_{04}g_{12}g_{20} +$$
$$90g_{02}g_{11}f_{11}^2g_{30} + 72g_{20}f_{12}f_{11}g_{30} - 99f_{13}g_{02}g_{20}g_{11} - 909g_{31}f_{11}g_{20}g_{02} + 90f_{03}f_{22}g_{20} +$$
$$27f_{41}g_{20}g_{11} - 15f_{22}g_{11}^2f_{11} - 81g_{20}^2f_{11}g_{13} - 297f_{31}g_{02}f_{12} + 207f_{11}g_{40}g_{21} - 18f_{13}f_{11}g_{20}g_{11} +$$
$$63g_{04}g_{12}g_{11} + 12g_{02}g_{11}^3g_{12} - 288f_{12}g_{02}^2g_{20}^2 - 18g_{20}f_{03}f_{11}^2g_{11} - 45f_{02}f_{11}^2g_{20}^3 + 420f_{12}g_{02}^2f_{02}^2 +$$
$$18f_{13}f_{11}^2g_{11} + 99f_{02}g_{22}g_{11}^2 + 210g_{02}f_{02}^3f_{21} + 72g_{31}g_{11}g_{21} + 105f_{02}f_{12}^2f_{11} - 87f_{12}g_{20}^2g_{11}^2 -$$
$$261g_{20}^2g_{21}f_{21} + 270f_{02}g_{02}^3f_{21} + 756g_{12}g_{02}g_{11}g_{20}^2 + 180g_{14}f_{02}f_{11} - 12g_{30}g_{11}^3f_{11} + 189f_{04}f_{03}f_{11} +$$
$$123f_{02}g_{21}g_{11}f_{21} - 216g_{02}^2f_{03}g_{21} - 72f_{11}g_{22}g_{20}g_{11} + 9f_{02}g_{20}f_{11}^2g_{11}^2 + 156f_{02}g_{21}g_{11}g_{12} -$$
$$255g_{20}f_{02}g_{11}^2g_{12} - 231f_{02}^2g_{11}f_{11}g_{12} - 198f_{22}f_{21}g_{02} + 1080g_{02}^2g_{21}g_{20}^2 + 36g_{12}f_{21}g_{21} +$$
$$270g_{12}^2f_{02}g_{02} + 1665g_{20}^3g_{11}g_{30} - 93g_{20}f_{02}g_{11}^2f_{21} - 72g_{02}^2g_{20}g_{11}^3 + 24f_{03}g_{02}g_{11}^3 + 1134g_{20}^2g_{11}^2g_{02}f_{02} +$$
$$2f_{11}f_{02}g_{11}^4 + 885g_{11}g_{31}f_{02}g_{02} - 27f_{21}f_{02}f_{11}g_{12} - 27f_{11}^2g_{40}f_{02} + 135f_{03}g_{20}g_{13} + 360f_{03}g_{11}g_{20}^3 -$$
$$603g_{20}g_{11}f_{11}g_{40} + 504g_{02}g_{20}^2f_{03}g_{11} - 81f_{04}f_{11}^2g_{02} + 66g_{11}^2g_{02}f_{22} + 108g_{02}f_{03}g_{13} + 252f_{04}f_{02}g_{11}g_{02} -$$
$$135f_{11}^2g_{20}^3g_{11} + 210f_{22}f_{02}^2g_{02} + 228f_{12}g_{02}^2f_{02}g_{11} - 117g_{20}g_{11}g_{02}f_{31} + 342f_{32}g_{20}g_{02} -$$
$$9f_{22}g_{20}^2f_{11} + 504f_{11}g_{02}g_{41} + 36f_{11}g_{02}f_{32} + 81f_{21}g_{20}^2f_{11}f_{02} + 54f_{04}g_{12}f_{11} + 9f_{04}g_{11}^2f_{11} +$$
$$9f_{21}g_{11}g_{22} + 81f_{03}g_{22}g_{11} - 504f_{04}g_{21}f_{02} + 18f_{02}g_{11}g_{13}f_{11} - 72g_{30}g_{21}f_{11}^2 + 135g_{21}f_{11}^2g_{20}^2 +$$
$$18f_{11}^2f_{12}g_{20}^2 - 1125g_{30}g_{21}g_{20}^2).$$

The computation of Lyapunov quantities by two different analytic methods with applying the modern software tools of symbolic computation permits us to show that the formulas, obtained for Lyapunov quantities, are correct.

APPENDIX 2 COMPUTATION OF LYAPUNOV QUANTITIES FOR LIENARD EQUATION IN MATLAB

```
clear all
syms x y h t 'real'
syms g11 g21 g31 g41 g51 g20 g30 g40 g50 g60 g70 'real'

 Nfg = 7; %System degree = 2m+1 for the general expression of L_m
 gxy = (g11*x+g21*x^2+g31*x^3+g41*x^4+g51*x^5)*y;
 gxy = gxy+g20*x^2+g30*x^3+g40*x^4+g50*x^5+g60*x^6+g70*x^7

 xt_s(1:Nfg-1) = 0*h; yt_s(1:Nfg-1) = 0*h; xth_s = 0*t; yth_s = 0*t;
 for n=1:Nfg
     xt_s(n) = sym(['xt_',int2str(n)],'real');  xth_s = xth_s + xt_s(n)*h^n;
     yt_s(n) = sym(['yt_',int2str(n)],'real');  yth_s = yth_s+ yt_s(n)*h^n;
 end

 NL = 7; % =2m+1 for L_m
 sT_h_cur = 0;
 for i = 1:NL-1
   sT_h(i,1) = sym(['T',int2str(i)],'real');  sT_h_cur = sT_h_cur + sT_h(i,1)*h^i;
 end;

 ugt(1:Nfg) = 0*t; xt(1:Nfg)=0*t; yt(1:Nfg)=0*t;
 xt(1) = -sin(t); yt(1) = cos(t); xt_cur = xt(1)*h; yt_cur = yt(1)*h;
 for i=2:NL
    ugt_s = subs(diff(subs(gxy, [x y], [xth_s yth_s]),h,i)/factorial(i),h,0);
    ugt(i) =  subs(ugt_s, [xt_s yt_s], [xt yt]);
    uIt = diff(ugt(i),t);
    Iucos = int(cos(t)*uIt,t); Iucos_t0 = (Iucos - subs(Iucos,t,0));
    Iusin = int(sin(t)*uIt,t); Iusin_t0 = (Iusin - subs(Iusin,t,0));
    ug0 = subs(ugt(i),t,0);
    xt(i) = simplify(cos(t)*ug0+Iucos_t0*cos(t) + Iusin_t0*sin(t)-ugt(i));
    yt(i) = simplify(sin(t)*ug0+Iucos_t0*sin(t) - Iusin_t0*cos(t));
    xt_cur = xt_cur + xt(i)*h^i; yt_cur = yt_cur + yt(i)*h^i;
 end;

 xh_cur = subs(xt_cur,t,2*pi);
 for k = 1:NL
     xh_cur = xh_cur + subs(diff(xt_cur,k,t),t,2*pi)*sT_h_cur^k/factorial(k);
 end;
 for k = 1:NL
     xh(k,1) = subs(diff(xh_cur,k,h)/factorial(k),h,0);
 end;

 xh_temp = xh; T_cur = 0;
 for k = 2:NL
     T(k-1,1) = solve(xh_temp(k,1),sT_h(k-1,1));
     T_cur = T_cur + T(k-1,1)*h^(k-1);
     xh_temp = subs(xh_temp,sT_h(k-1,1),T(k-1,1));
 end;

 yh_cur = subs(yt_cur,t,2*pi);
 for k = 1:NL
     yh_cur = yh_cur + subs(diff(yt_cur,k,t),t,2*pi)*T_cur^k/factorial(k);
 end;
 for k = 1:NL
     yh(k,1) = subs(diff(yh_cur,k,h)/factorial(k),h,0);
 end;
 yh = factor(yh);

 % L_m = yh(2m+1,1) if L_{k<m}=0
 L1 = factor(yh(3,1))
 g21_s = solve(L1,'g21')
```

```
L2 = factor(subs(yh(5,1),'g21',g21_s))
g41_s = factor(solve(L2,'g41'))
L3 = factor(subs(yh(7,1),'g41',g41_s))
```

REFERENCES

[Abramovitch, 2002] Abramovitch D. [2002] "Phase-Locked Loop: A control Centric. Tutorial" *Proceedings of the 2002 ACC*.

[Adrianova, 1998] Adrianova L.Ya. [1998] *Introduction to Linear systems of Differential Equations* (American Mathematical Society, Providence, Rhode Island).

[Aleksenko, 2004] Aleksenko A. [2004] *Digital engineering* [in Russian] (Moscow: Unimedstyle).

[Andronov *et al.*, 1966] Andronov A.A., Vitt E.A. & Khaiken S.E. [1966] *Theory of Oscillators* (Pergamon Press, Oxford).

[Anishchenko *et al.*, 2002] Anishchenko V.S., Astakhov V.V., Neiman A.B., Vadivasova T.E. & Schimansky-Geier L. [2002] *Nonlinear Dynamics of Chaotic and Stochastic Systems* (Springer, New York).

[Anosov *et al.*, 1997] Anosov D.V., Aranson S.Kh., Arnold V.I., Bronshtein I.U., Grines V.Z. & Il'yashenko Yu.S. [1997] *Ordinary Differential Equations and Smooth Dynamical Systems* (Springer, New York).

[Bacciotti & Rosier, 2005] Bacciotti A. & Rosier L. [2005] *Liapunov functions and stability in control theory, 2nd Edition* (Springer).

[Barabanov, 2005] Barabanov E.A. [2005] "Singular exponents and regularity criterion for linear differential systems" [in Russian], *Diff. uravneniya* **41**, 2, pp. 147–157.

[Bautin & Leontovich, 1976] Bautin N.N. & Leontovich E.A. [1976] *Methods and procedures for qualitative study of dynamical systems on plane* [in Russian] (M.: Nauka).

[Bautin, 1952] Bautin N. [1952] "On the number of limit cycles arising with the variation of the coefficients from an equilibrium point of focus or center type" [in Russian], *Mat. Sbornik*, 30(72), pp. 181–196 [*AMS* [1954] as Translation Number 100, 396–413; reprint, AMS Transl. (1) 5, pp. 396–413].

[Bautin, 1949] Bautin N.N. [1949] *Behavior of Dynamic Systems Near the Boundaries of the Stability Region* [in Russian], (Gostekhizdat, Moscow-Leningrad).

[Bellman & Cooke, 1963] Bellman R. & Cooke K.L. [1963] *Differential-difference equations* (Academic Press, New York-London).

[Bellman, 1953] Bellman R. [1953] *Stability theory of differential equations* (New York, McGraw-Hill).

[Bennet *et. al.*, 2005] Bennet M., Schatz M.F., Rockwood H.& Wiesenfeld K. [2005] "Huygens' clock", *Royal society* **1**, 1, pp. 1–18.

[Best, 2003] Best R.E. [2003] *Phase-Lock Loops: Design, Simulation and Application* 5^{ed} (McGraw Hill).

[Blekhman, 1988] Blekhman I.I. [1988] *Synchronization in Science and Technology* (ASME, New York).

[Blows & Perko, 1990] Blows T.R. & Perko L.M. [1990] *Bifurcation of Limit Cycles from Centers* (Preprint Univ. N. Arizona).

[Boichenko *et al.*, 2005] Boichenko V.A., Leonov G.A. & Reitmann V. [2005] *Dimension Theory for Ordinary Differential Equations* (Stuttgart: Teubner).

[Bromberg, 1986] Bromberg V.P. [1986] *Matrix method in the theory of relay and pulse regulation* [in Russian] (Nauka, Moscow).

[Bylov *et al.*, 1966] Bylov B.F., Vinograd R.E., Grobman D.M. & Nemytskii V.V. [1966] *The Theory of Characteristic Exponents and its Applications to Problems of Stability* [in Russian] (Nauka, Moscow).

[Cesari, 1959] Cesari L. [1959] *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations* (Berlin: Springer).

[Cesari, 1959] Cesari L. [1959] *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations* (Springer, Berlin).

[Chavarriga & Grau, 2003] Chavarriga J. & Grau M. [2003] "Some open problems related to 16th Hilbert problem", *Sci. Ser. A Math. Sci. (N.S.)*, 9, pp. 1–26.

[Cherkas, 1976] Cherkas L.A. [1976] "Number of limit cycles of an autonomous second-order system", *Differential Equations*, 5, pp. 666–668.

[Chetaev, 1948] Chetaev N.G. [1948] "Some problems of stability and instability for nonregular systems" [in Russian], *Prikl. Math. i Mech.* **12**, 6, pp. 639–642.

[Chetaev, 1955] Chetaev N.G. [1955] *Stability of motion* [in Russian] (Gostekhizdat, Moscow).

[Chetaev, 1990] Chetaev N.G. [1990] *Stability of Motion* [in Russian] (Nauka, Moscow).

[Christopher & Li, 2007] Christopher C. & Li Ch. [2007] *Limit cycles of differential equations* Advanced Courses in Mathematics. CRM Barcelona. (Basel: Birkhauser Verlag).

[Coppel, 1965] Coppel W.A. [1965] *Stability and asymptotic behavior of differential equations* (D.C. Heath, Boston).

[Coppel, 1978] Coppel W.A. [1978] *Dichotomies in Stability Theory* (Springer-Verlag).

[Davies & James, 1966] Davies T.V. & James E.M. [1966] *Nonlinear differential equations* (Addison-Wesley).

[Demidovich, 1967] Demidovich B.P. [1967] *Lectures on mathematical theory of stability* [in Russian] (M.: Nauka).

[Demidovich, 1969] Demidovich V.B. [1969] "Stability criterion for difference equations" [in Russian], *Diff. uravneniya* **5**, 7, pp. 1247–1255.

[Egan, 2000] Egan W.F. [2000] *Frequency Synthesis by Phase Lock* 2^{ed} (New York: John Wiley & Sons).

[Filippov, 1988] Filippov A.F. [1988] *Differential Equations with Discontinuous Right Hand Sides* (Kluwer, Dordrecht).

[Frommer, 1934] Frommer M. [1934] "Uber das Auftreten von Wirbeln und Strudeln (geschlossener und spiraligerIntegralkurven) in der Umgebung Rationalre Unbestimmtkeitssellen", *Math. Annalen* 109, pp. 395–424.

[Gardner, 2005] Gardner F. [2005] *Phase–lock techniques* 2^{ed} (New York: John Wiley & Sons).

[Gasull *et al.*, 1997] Gasull A., Guillamon A. & Manosa V. [1997] "An explicit expression of the first Liapunov and period constants with applications", *J. Math. Anal. Appl.*, 211, pp. 190–212.

[Gasull *et al.*, 1999] Gasull A., Guillamon A. & Manosa V. [1999] "An analytic-numerical method for computation of the Liapunov and period constants derived from their algebraic structure", *SIAM J. Numer. Anal.*, 36, No.4, pp. 1030–1043.

[Gayschun, 2001] Gayschun I.V. [2001] *Systems with discrete time* [in Russian] (Inst. matemat. AN Belarusi, Minsk).

[Gine, 2007] Gine J. [2007] "On some problems in planar differential systems and Hilbert's 16th problem", *Chaos, Solutions and Fractals*, Vol. 31, pp. 1118–1134.

[Goldhirsch *et al.*, 1987] Goldhirsch I., Sulem P.L. & Orszag S.A. [1987] "Stability and Lyapunov stability of dynamical systems: a differential approach and numerical method", *Physica* **D 27**, pp. 311–337.

[Hahn, 1967] Hahn W. [1967] *Stability of motion* (Springer-Verlag, Berlin-Heidelberg).

[Halanay, 1966] Halanay A. [1966] *Differential Equation* (Academic Press, New York–London).

[Hartman, 1984] Hartman P. [1984] *Ordinary differential equations, Second edition* (BirkHauser).

[Heagy & Hammel, 1994] Heagy J.F. & Hammel S.M. [1994] "The birth of strange nonchaotic attractors", *Physica* **D 70**, N1,2, pp. 140–153.

[Huygens, 1669] Huygens C. [1669] "Instructions concerning the use of pendulum-watches, for finding the Longitude at Sea", *Philo.T rans. Royal Soc.* 4, No 47, 937.

[Huygens 1986] Huygens C. (translated by R. Blackwell) *Christiaan Huygens' The pendulum clock or geometrical demonstrations concerning the motion of pendulum as applied to clocks* (Ames, IA: Iowa State University Press).

[Katok & Hasselblat, 1995] Katok A. & Hasselblatt B. [1995] *Introduction to the modern theory of dynamical systems* (Cambridge univ. press).

[Kroupa, 2003] Kroupa V. [2003] *Phase Lock Loops and Frequency Synthesis* (New York: John Wiley & Sons).

[Kung, 1988] Kung S. [1988] *VLSI Array Processors* (New York: Prentice Hall).

[Kuznetsov *et al.*, 2008[1]] Kuznetsov N.V., Leonov G.A. & Seledzhi S.M. [2008] "Phase Locked Loops Design And Analysis", *Int. Conference on Informatics in Control, Automation and Robotics, Proceedings*, pp. 114–118.

[Kuznetsov *et al.*, 2008[2]] Kuznetsov N.V., Kudryashova E.V., Leonov G.A. & P. Neittaanmäki [2008] "Large cycles in a series of computer experiments", *Abstracts of Int. conference "Space, astronomy and programming"*, p. 81.

[Kuznetsov & Leonov, 2008[1]] Kuznetsov N.V. & Leonov G.A. [2008] "Computation of Lyapunov quantities", *6th EUROMECH Nonlinear Dynamics Conference* (http://lib.physcon.ru/?item=1802)

[Kuznetsov & Leonov, 2008[2]] Kuznetsov N.V. & Leonov G.A. [2008] Limit cycles and strange behavior of trajectories in two dimension quadratic systems, *Journal of Vibroengineering* [in print].

[Kuznetsov *et al.*, 2007[1]] Kuznetsov N.V., Leonov G.A. & Seledzhi S.M. [2007] "Global stability of phase-locked loops", *3rd Int. IEEE conference on Physics and Control* (http://lib.physcon.ru/?item=1192).

[Kuznetsov *et al.*, 2007[2]] Kuznetsov N.V., Leonov G.A., Nijmeijer H. & Pogromsky A. [2007] "Synchronization of two metronomes", *3rd IFAC Workshop on Periodic Control System* (http://lib.physcon.ru/?item=1430).

[Kuznetsov & Leonov, 2007] Kuznetsov N.V. & Leonov G.A. [2007] "Computation of the first Lyapunov quantity for the second-order dynamical system", *Third IFAC Workshop Periodic Control System* (http://www.ifac-papersonline.net).

[Kuznetsov *et al.*, 2006[1]] Kuznetsov N.V., Kuznetsova O.A. & Leonov G.A. [2006] "Uniform stability and Perron effects" [in Russian], *Diff. Uravneniya. Izvestiya RAEN* N11, pp. 125–131.

[Kuznetsov *et al.*, 2006[2]] Kuznetsov N.V., Leonov G.A. & Seledzhi S.M. [2006] "Analysis of phase-locked systems with discontinuous characteristics of the phase detectors", *Preprints of 1st IFAC conference on Analysis and control of chaotic systems*, pp. 127–132.

[Kuznetsov & Leonov, 2005[1]] Kuznetsov N.V. & Leonov G.A. [2005] "Criterion of stability by the first approximation for nonlinear discrete systems", (part II), *Vestnik St.Petersburg Univ., Math.* **38**, N3, pp. 30–42.

[Kuznetsov & Leonov, 2005[2]] Kuznetsov N.V. & Leonov G.A. [2005] "Criterion of stability by the first approximation for nonlinear discrete systems", (part I), *Vestnik St.Petersburg Univ., Math.* **38**, N2, pp. 55–63.

[Kuznetsov & Leonov, 2005[3]] Kuznetsov N.V. & Leonov G.A. [2005] "On stability by the first approximation for discrete systems", *Proceedings of 2-th Int. conference of Physics and Control*, pp. 596–599.

[Kuznetsov & Leonov, 2005[4]] Kuznetsov N.V. & Leonov G.A. [2005] "Criteria of Krasovsky instability for nonstationary linearization", *Thesis of conference Dynamical System Modeling And Stability Investigation*, p. 72.

[Kuznetsov & Leonov, 2001] Kuznetsov N.V. & Leonov G.A. [2001] "Counterexample of Perron in the discrete case" [in Russian], *Diff. Uravneniya. Izvestiya RAEN*, 5, p. 71.

[Lakshmikantham *et al.*, 1989] Lakshmikantham V., Bainiv D.D. & Simeonov P.S. [1989] *Theory of impulsive differential equations* (World Scientific, Singapore).

[Lapsley *et al.*, 1997] Lapsley P., Bier J., Shoham A. & Lee E. [1997] *DSP Processor Fundamentals Architecture and Features* (New York: IEE Press).

[LaSalle, 1980] LaSalle J.P. [1980] *The Stability and Control of Discrete Processes* (Springer-Verlag).

[Lefschetz, 1957] Lefschetz S. [1957] *Differential Equations: Geometric Theory* (New York: Interscience Publishers).

[Leonov & Kuznetsova 2008] Leonov G.A. & Kuznetsova O.A. [2008] "Calculation of the first five Lyapunov quantities for the Lienard system", *Doklady Akademii Nauk* [in print].

[Leonov *et al.*, 2008] Leonov G.A., Kuznetsov N.V. & Kudryashova E.V. [2008] "Cycles of two-dimensional systems. Computations, proofs, experiments." *Vestnik St.Petersburg Univ., Math.*, No.3 [in print].

[Leonov & Kuznetsov, 2007] Leonov G.A. & Kuznetsov N.V. [2007] "Time-Varying Linearization and Perron effects", *International journal of bifurcation and chaos*, Vol. 17, N4, pp. 1–29.

[Leonov & Seledzhi, 2002] Leonov G. & Seledzhi S. [2002] *Phase locked loops in array processors* [in Russian] (St.Petersburg: Nevsky dialekt).

[Leonov *et al.*, 1992] Leonov G., Reitmann V. & Smirnova V. [1992] *Nonlocal Methods for Pendulum-Like Feedback Systems* (Stuttgart, Leipzig: Teubner Verlagsgesselschaft).

[Leonov *et al.*, 1996] Leonov G., Ponomarenko D. & Smirnova V. [1996] *Frequency-Domain Methods for Nonlinear Analysis. Theory and Applications* (Singapore: World Scientific).

[Leonov, 2001[1]] Leonov G.A. [2001] *Mathematical Problems of Control Theory, An Introduction* (World Scientific, Singapore).

[Leonov, 2001[2]] Leonov G.A. [2001] "Lyapunov Dimension Formulas for Henon and Lorenz Attractors", *St.Petersburg Mathem. J.* **13**, 3, pp. 453–464.

[Leonov, 2002[1]] Leonov G.A. [2002[1]] "Instability in the First Approximation for Time-Dependent Linearization", *J. of Appl. Mathem. and Mech.* **66**, 2, pp. 323–325.

[Leonov, 2002[2]] Leonov G.A. [2002[2]] "Strange attractors and the classical theory of motion stability" [in Russian], *Uspekhi mekhaniki* **1**, 3, pp. 3–42.

[Leonov, 1998] Leonov G.A. [1998] "On Stability in the First Approximation", *J. of Appl. Mathem. and Mech.* **62**, 4, pp. 511–517.

[Leonov, 2003] Leonov G.A. [2003] "A Modification of Perron's Counterexample", *Differential Equations* **39**, 11, pp. 1651–1652.

[Leonov, 2004] Leonov G.A. [2004] "Criteria of Instability in the First Approximation for Time – Dependent Linearizations", *J. of Appl. Mathem. and Mech.* **68**, pp. 827–838.

[Leonov, 2008] Leonov G.A. [2008] *Strange attractors and classical stability theory* [in English] (St. Petersburg University Press, Russia).

[Li, 2003] Li J. [2003] Hilbert's 16th problem and bifurcation of planar polynomial vector fields, *Int. J. Bifurcation Chaos* **13**, pp. 47–106.

[Lindelof E., 1894] Lindelof E. [1894] "Sur l'application des methodes d'approximations successives a l'etude des integrales reeles des equations differentielles ordinaire", *J. Math. Pures AppL*, 10 (4), 117–128 [II. 1].

[Lindsey & Chie, 1981] Lindsey W. & Chie C. [1981] "A Survey of Digital Phase Locked Loops", *Proceedings of the IEEE*.

[Lindsey, 1972] Lindsey W. [1972] *Synchronization systems in communication and control* (New Jersey: Prentice-Hall).

[Lloyd & Pearson, 1997] Lloyd N.G. & Pearson J. [1997] "Five limit cycles for a simple cubic system", *Publicacions Matem'atiques*, Vol. 41. pp. 199–208.

[Lloyd, 1988] Lloyd N.G. [1988] Limit cycles of polynomial systems – some recent developments. In *New Direction in Dynamical Systems.* (Cambridge University Press) pp. 192–234.

[Luca & Vleck, 2002] Luca D. & Van Vleck E. [2002] "Lyapunov and Other Spectra: A Survey", *Collected Lectures on the Preservation of Stability under Discretization*, SIAM, pp. 197–218.

[Luca *et al.*, 1997] Luca D., Russell R.D. & Van Vleck E. [1997] "On the computation of Lyapunov exponents for continuous dynamical systems", *SIAM J. Numer. Anal.* **34**, pp. 402–423.

[Lyapunov, 1892] Lyapunov A.M. [1892] *Stability of Motion* (Academic Press, New York).

[Lynch, 2005] Lynch S. [2005] "Symbolic computation of Lyapunov quantities and the second part of Hilbert's sixteenth problem" *Differential Equations with Symbolic Computations*, pp. 1–26, Wang, Dongming; Zheng, Zhiming (Eds.), *Series: Trends in Mathematics* (Springer).

[Malkin, 1966] Malkin I.G. [1966] *Theory of motion stability* [in Russian] (Nauka, Moscow).

[Marsden & McCracken, 1976] Marsden J. & McCracken M. [1976] *Hopf bifurcation and its applications* (New York: Springer).

[Massera, 1956] Massera J.L. [1956] "Contributions to Stability Theory", *Ann. Math.*, **64**, pp. 182–206; **68**, p. 202.

[Maxwell *et al.*, 1949] Maxwell D.K., Vyshnegradskii I.A. & Stodola A. [1949] *Automatic Control Theory, Linearized Problems* (AN SSSR, Moscow).

[Maxwell, 1868] Maxwell J. [1868] "On the governors", *Proceedings of the Royal Society*, **270**, p. 16.

[Moon, 1987] Moon F. [1987] *Chaotic Vibrations* (John Wiley, New York).

[Neimark & Landa, 1992] Neimark Yu.I. & Landa P.S. [1992] *Stochastic and Chaotic Oscillations* (Kluwer, Dordrecht).

[Oud *et. al.*, 2006] Oud W., Nijmeijer H. & Pogromsky A. [2006] "Experimental results on Huygens synchronization", *Proceedings of 1st IFAC Conference on Analysis and Control of Chaotic Systems*.

[Pantaleone, 2002] Pantaleone J. [2002] "Synchronization of metronomes", *Am. J. Phys.* **70**, 10, pp. 992–1000.

[Perko, 1990] Perko L.M. [1990] "Global Families of Limit Cycles of Planar Analytic Systems", *Trans. Amer. Math. Soc*, Vol. 322, pp. 627–656.

[Perron, 1930] Perron O. [1930] "Die Stabilitatsfrage bei Differentialgleichungen", *Mathematische Zeitschrift* **bd. 32**, 5, pp. 702–728.

[Persidskii, 1947] Persidskii K.P. [1947] "Characteristic numbers of differential equations" [in Russian], *Izv. AN Kazakhskoi SSR, ser.mat.i mekh.*, **1**.

[Pogromsky *et. al.*, 2005] Pogromsky A.Y., Belykh V.N. & Nijmeijer H. [2005] "A study of controlled synchronization of Huijgens pendula", *Group Coordination and Cooperative Control* Editors: K.Y.Pettersen, J.T.Gravdahl, and H.Nijmeijer, pp. 205–216 (Springer Verlag).

[Poincare, 1885] Poincare H. [1885] "Memoire sur les courbes definies par lesequations diffeentielles", *J. de. Mathematiques Pures et Appliquees*, (**4**) Vol. 1, pp. 167–244.

[Poincare, 1947] Poincare A. [1947] "Sur les courles definies par les equations differentielles", *Journal de math.* **3**, 7, pp. 375–422, (1882) 8, pp. 251–296, (1885) **4**, 1, pp. 167–244, (1886) 2, pp. 151–217.

[Razavi, 2003] Razavi B. [2003] *Phase-Locking in High-Performance Systems: From Devices to Architectures* (New York: John Wiley & Sons).

[Reyn, 1994] Reyn J.W. [1994] *A bibliography of the qualitative theory of quadratic systems of differential equations in the plane. Third edition. Report 94-02* (Delft University of Technology).

[Roussarie, 1998] Roussarie R. [1998] *Bifurcations of planar vector felds and Hilbert's sixteenth problem. Progress in mathematics. V.164.* (Basel; Boston; Berlin: Birkhauser).

[Russel *et al.*, 1980] Russel D.A., Hanson J.D. & Ott E. [1980] "Dimension of Strange Attractors", *Phys. Rev. Lett.* **45**, 14, pp. 1175–1178.

[Ryabov, 2002] Ryabov V.B. [2002] "Using Lyapunov exponents to predict the onset of chaos in nonlinear oscillators", *Physical Review E* **66**, pp. 16214–16231.

[Sansone & Conti 1964] Sansone G. & Conti R. [1964] *Non-linear differential equations* (Pergamon Press).

[Schuster, 1984] Schuster H.G. [1984] *Deterministic Chaos* (Physik — Verlag, Weinheim).

[Serebryakova, 1959] Serebryakova N. [1959] "Behavior of dynamical system with one degree of freedom near the points of boundaries of domain of stability, where a safe boundary goes to the dangerous one" [in Russian] *Izvestiya AN SSSR. Division of technical sciences, mechanics and engineering*, N2, pp. 178–182.

[Shi, 1980] Shi S.L. [1980] "A concrete example of the existence of four limit cycles for plane quadratic systems", *Sci. Sinica* 23. pp. 153–158.

[Shilnikov *et al.*, 2001] Shilnikov L., Turaev D. & Chua L. [2001] *Methods of Qualitative Theory in Nonlinear Dynamics: Part 2* (World Scientific).

[Smith, 1999] Smith S. [1999] *The Scientist and Engineers Guide to Digital Dignal Processing* (San Diego: California Technical Publishing).

[Solonina *et al.*, 2000] Solonina A., Ulahovich D. & Jakovlev L. [2000] *The Motorola Digital Signal Processors* [in Russian] (St. Petersburg: BHV).

[Temam, 1988] Temam R. [1988] *Infinite-Dimensional Dynamical Systems in Mechanics and Physics* (Springer-Verlag, New York).

[Ugrumov, 2000] Ugrumov E. [2000] *Digital engineering* [in Russian] (St. Petersburg: BHV).

[Vinograd, 1954] Vinograd R.E. [1954] *Uspekhi matematicheskih nauk* [in Russian], **9**, 2, pp. 129–136.

[Viterbi, 1966] Viterbi A. [1966] *Principles of coherent communications* (New York: McGraw-Hill).

[Vyshnegradskii, 1877] Vyshnegradskii I.A. [1877] "Direct-acting regulators" [in Russian], *Izv. Sankt-Peterburgskogo tekhnologicheskogo instituta*, pp. 21–62.

[Wasow, 1965] Wasow W. [1965] *Asymptotic expansions for ordinary differential equations* (Interscience publ., New York).

[Willems, 1970] Willems J.L. [1970] *Stability theory of dynamical systems* (John Willey and Sons, Inc., New York).

[Wu *et al.* 1994] Wu X. et al. [1994] "Nonlinear dynamical behavior of the limited explodator in a CSTR under square wave perturbation of the flow rate", *Physica* D 74, 1,2, pp. 74–89.

[Yakubovich *et al.*, 2004] Yakubovich V.A., Leonov G.A. & Gelig A.Kh. [2004] *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities* (World Scientific, Singapore).

[Yoshizawa, 1966] Yoshizawa T. [1966] *Stability theory by Liapunov's second method* (The Mathematical Society of Japan, Tokyo).

[Yu & Chen, 2008] Yu P. & Chen G. [2008] Computation of focus values with applications, *Nonlinear Dynamics*, **51**(3), pp. 409–427.

[Yu & Han, 2005] Yu P. & Han M. [2005] "Twelve limit cycles in a cubic case of the 16th Hilbert problem" *Int. J. Bifurcations and Chaos* 15(7), pp. 2191–2205.